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# Escape and absorption probabilities for obliquely reflected Brownian motion in a quadrant

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**Abstract**

We consider an obliquely reflected Brownian motion $Z$ with positive drift in a quadrant stopped at time $T$, where $T := \inf\{t > 0 : Z(t) = (0, 0)\}$ is the first hitting time at the origin. Such a process can be defined even in the non-standard case in which the reflection matrix is not completely-$\mathcal{S}$. We show in this case that the process has two possible behaviors: either it tends to infinity or it hits the corner (origin) in finite time. Given an arbitrary starting point $(u, v)$ in the quadrant, we consider the escape (resp. absorption) probabilities $\mathbb{P}_{(u,v)}[T = \infty]$ (resp. $\mathbb{P}_{(u,v)}[T < \infty]$). We establish the partial differential equations and the oblique Neumann boundary conditions which characterize the escape probability and provide a functional equation satisfied by the Laplace transform of the escape probability. Asymptotics for the absorption probability in the simpler case in which the starting point in the quadrant is $(u, 0)$ are then given. We proceed to show a geometric criterion on the parameters which characterizes the case in which the absorption probability has a product form and is exponential. We call this new criterion the *dual skew symmetry* condition due to its natural connection with the skew symmetry condition for the stationary distribution. We then obtain an explicit integral expression for the Laplace transform of the escape probability and conclude by presenting exact asymptotics for the escape probability at the origin.



*MSC:* primary 60J60; 60K25; secondary 90B22; 30D05

*Keywords:* Escape and absorption probability; Obliquely reflected Brownian motion in a quadrant; Functional equation; Carleman boundary value problem; Neumann's condition; Asymptotics

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# 1. Introduction

## 1.1. Model and goal

Let $Z(t) = (Z_1(t), Z_2(t))$ be a reflected Brownian motion (RBM) in the quadrant, starting from the point $(u, v)$, with positive drift $\mu = (\mu_1, \mu_2)$; that is, $\mu_1 > 0, \mu_2 > 0$. See below for references and motivation. The covariance matrix is $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ with $|\rho| < 1$ and the reflection matrix is $\begin{pmatrix} 1 & -r_2 \\ -r_1 & 1 \end{pmatrix}$. We further assume that

$$r_1 > 0, \ r_2 > 0 \text{ and } 1 \leqslant r_1 r_2. \tag{1}$$

See Fig. 1 for a representation of the parameters. We define this reflected process up to the first hitting time $T$ of the corner, defined as

$$T := \inf\{t > 0 : Z(t) = 0\}.$$

For $t \leqslant T$, this process may be written as

$$\begin{cases} Z_1(t) := u + W_1(t) + \mu_1 t + l_1(t) - r_2 l_2(t), \\ Z_2(t) := v + W_2(t) + \mu_2 t - r_1 l_1(t) + l_2(t), \end{cases} \tag{2}$$

where $l_1(t)$ (resp. $l_2(t)$) is a local time on the vertical (resp. horizontal) axis and is a continuous non-decreasing process starting from 0 which increases only when $Z_1(t) = 0$ (resp. $Z_2(t) = 0$). Under condition (1), when $t > T$, that is after that the process $Z$ hits the corner, the process is no longer defined by (2) for reasons of convexity. In lieu, for $t > T$, we define $Z(t) = (0, 0)$ and say that the process is absorbed when $T < \infty$. Further details on the existence and uniqueness of this process will be given in Section 1.2.

The objective of the present paper is to study the probability of escape to infinity for a process starting from $(u, v)$. We denote this probability as

$$\mathbb{P}_{(u,v)}[T = \infty].$$

The corresponding absorption probability at the origin is $\mathbb{P}_{(u,v)}[T < \infty] = 1 - \mathbb{P}_{(u,v)}[T = \infty]$.

Since its introduction in the eighties by Harrison, Reiman, Varadhan and Williams [25,26, 42,44,45], reflected Brownian motion in the quarter plane has received significant attention. Recurrence and transience of obliquely reflected Brownian motion were examined in [29,44]. The process has also been considered in planar domains [24,27] as well as in general dimensions in orthants [26,41,46]. The stationary distribution of obliquely reflected Brownian motion has been studied in [9,10,12,21,31] and its Green's functions have been studied in [18]. The roughness of its paths were studied in [32]. Obliquely reflected Brownian motion has played an important role in applications concerning heavy traffic approximations for open queueing networks [22,39]. It has also been utilized in queueing models as diffusion approximations for tandem queues [33,34,37].

There are several possible interpretations in insurance risk of models involving reflected Lévy processes in a quadrant [1,4,30]. For example, suppose there are two funds, each of whose free surplus is modeled by a Cramér–Lundberg process, and which have the following agreement: a deficit in one fund is instantly covered by the other fund, with ruin occurring when neither company can cover the deficit of the other. In the case of the present problem, the absorption probability may be interpreted as the probability of ruin; the escape probability may be interpreted as the probability of survival and infinite capital expansion. The aforementioned

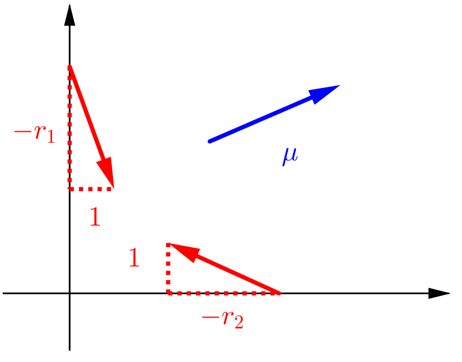



**Fig. 1.** Reflection vectors and drift.

process also arises in the study of queueing models as diffusion approximations for some Lévy tandem queues [7,17,43].

Previous works [3,13,16,17,21] have adapted an analytic method initially developed for random walks by Fayolle and Iasnogorodski [14] and Malyshev [36] for studying obliquely reflected Brownian motion. In particular, [17] focuses on a non-standard regime in which the reflected process escapes to infinity along one of the axes. Some of the techniques employed to solve the present problem are inspired by [17].

### *1.2. Definition of the process given in (2)*

Brownian motion in a quadrant with oblique reflection is usually defined as a process which behaves as a standard Brownian motion in the interior of the quadrant. It reflects instantaneously on the edges with constant direction and the amount of time spent at the origin has Lebesgue measure zero ([42]). Such a process is defined as a solution of a submartingale problem [42]. An interesting case arises when the process is a semimartingale reflecting Brownian motion (SRBM). Reiman and Williams [40] showed that a necessary condition for the process to be an SRBM is for the reflection matrix to be completely-$\mathcal{S}$.[1] Taylor and Williams [41] showed that this condition was also sufficient for the existence of an SRBM, which is unique in law.

Due to condition (1), the reflection matrix of the process in (2) is not completely-$\mathcal{S}$. The process indeed is not a standard SRBM as it can be trapped at the origin. Nonetheless, it is possible to define this absorbed process up to the stopping time $T$. The existence and uniqueness as a solution of a submartingale problem for the absorbed process is given in [42, §2.1, Thm 2.1]. Further, in Taylor and Williams [41, §4.2 and §4.3], the existence and uniqueness of an SRBM absorbed at the origin are proven without assuming that the reflection matrix is completely-$\mathcal{S}$.

### *1.3. From the quadrant to the wedge*

Franceschi and Raschel [21, Appendix] recently showed that studying reflected Brownian motion in a quadrant is equivalent to studying reflected Brownian motion in a wedge with angle

[1] A square matrix $R$ is said to be completely-$\mathcal{S}$ if for each principal sub-matrix $\widetilde{R}$ there exists $\widetilde{x} \geqslant 0$ such that $\widetilde{R}\widetilde{x} > 0$.

$\beta$, with identity covariance matrix, with two reflection angles $\delta$ and $\epsilon$, and with drift angle $\theta$ (see Fig. 2). The angles $\delta, \epsilon, \beta$ and $\theta$ (when the drift is nonzero) are in $(0, \pi)$ and are defined by

$$\tan\delta = \frac{\sin\beta}{-r_2+\cos\beta}, \quad \tan\epsilon = \frac{\sin\beta}{-r_1+\cos\beta}, \quad \tan\theta = \frac{\sin\beta}{\mu_1/\mu_2+\cos\beta}, \quad \cos\beta = -\rho. \tag{3}$$

The angles are equal to $\pi/2$ when the denominators of the tangents are equal to 0. Further, we denote $\alpha$ to be

$$\alpha := \frac{\delta+\epsilon-\pi}{\beta}. \tag{4}$$

Condition (1) is equivalent to $\delta+\epsilon-\beta \geqslant \pi$ (or equivalently $\alpha \geqslant 1$) and $\delta > \beta$, $\epsilon > \beta$.

### 1.4. The case of zero drift

The case of zero drift ($\mu = 0$) was treated by Varadhan and Williams [42]. In this case the absorption probability does not depend on the starting point. We recall from Varadhan and Williams [42, Thm 2.2] that

$$\mathbb{P}[T<\infty] = \begin{cases} 1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha \leqslant 0. \end{cases}$$

If $\alpha \leqslant 0$, the corner is not reached. If $0 < \alpha < 2$, the corner is reached but the amount of time spent by the process in the corner has Lebesgue measure zero. If $\alpha \geqslant 2$, the process reaches the corner and remains there. The previous properties remain valid in the case of zero drift. Under condition (1), the case of positive drift poses a new challenge, as $0 < \mathbb{P}_{(u,v)}[T<\infty] < 1$. We remark that condition (1) is equivalent to $\alpha \geqslant 1$.

### 1.5. Escape probability and stationary distribution of the dual process

Harrison [22] and Foddy [16] showed that the hitting time on one of the axes is inherently connected to the stationary distribution of a certain dual process. As the present article was nearing completion, it came to our attention that Harrison [23] extended the results from his earlier work [22] by introducing a dual RBM in the quadrant with drift $-\mu$ and reflection matrix

$$\begin{pmatrix} r_2 & -1 \\ -1 & r_1 \end{pmatrix},$$

where $1 < r_1 r_2$. This configuration of parameters is depicted in Fig. 3. This dual process has an explicit connection with the study of the escape probability. In particular, Harrison [23, Cor. 2] states that

$$\mathbb{P}_{(u,v)}[T=\infty] = \pi(\mathcal{S}(u,v)),$$

where $\pi$ is the stationary distribution of the dual process and $\mathcal{S}(u,v) := \{(u-r_2z_1+z_2, v+z_1-r_1z_2) \in \mathbb{R}_+^2 : (z_1,z_2) \in \mathbb{R}_+^2\}$ is a trapezoid as pictured in Fig. 3.

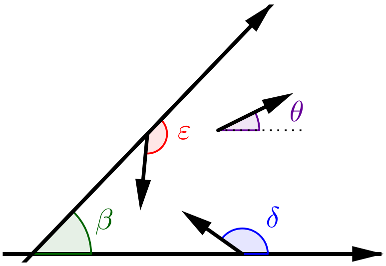



**Fig. 2.** Reflected Brownian motion in a wedge with angle $\beta$, reflection angles $\delta$ and $\epsilon$, and drift angle $\theta$.

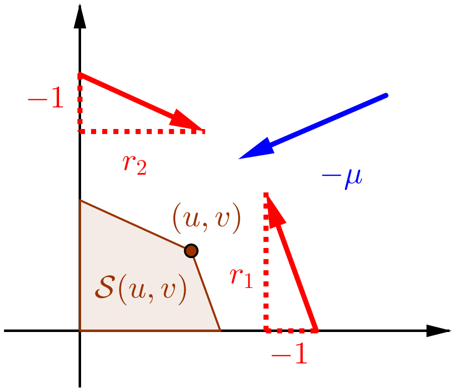



**Fig. 3.** Dual process parameters and trapezoid $\mathcal{S}(u, v)$ in brown.

### *1.6. Outline*

The remainder of this paper is organized as follows. In Section 2 we explore some general properties of the process of interest given in (2). This section's key result is Theorem 10, which states that the process has only two possible behaviors: either $T < \infty$, which means that the process is absorbed at the origin in finite time, or $T = \infty$, in which case the process escapes to infinity, namely $Z(t) \to \infty$ when $t \to \infty$. In Section 3 we present Proposition 11, which provides a partial differential equation characterizing the escape probability. Later in this section, we give Proposition 12, which provides a functional equation satisfied by the Laplace transform of the escape probability. In Section 4, we study the kernel of this functional equation and obtain asymptotic results for the absorption probability in the simpler case in which the starting point is $(u, 0)$ (Proposition 17). In Section 5, we find a geometric condition which characterizes the cases where the absorption probability has a product form and is exponential (Theorem 20). This result is reminiscent of the famous skew symmetry condition studied for invariant measures [25,28]. In Section 6, boundary value problem (BVP) satisfied by the Laplace transform of the escape probability (Proposition 22) is established. We continue with Theorem 30, which gives an explicit integral formula of this transform. In Section 7 exact asymptotics for the escape probability at the origin are obtained.

**In memory of Larry Shepp.** We dedicate this article in memory of our colleague, mentor, and friend, Professor Larry Shepp. Professor Shepp indelibly contributed to many areas of applied

probability, and one of the areas that interested him most concerned RBM in a quadrant as well as in a strip [24,27].

## 2. General properties of process $Z$

In this section we investigate a few key properties of the process given in (2). We prove three key results. The first is that if the starting point tends to infinity, then the probability that the process does not hit the origin tends towards 1 (Theorem 4). The second is that when the starting point tends to the origin, the probability that the process hits the origin in finite time tends towards 1 (Theorem 6). The third key result is that the process has only two possible behaviors: either $T < \infty$, which means that the process is absorbed at the origin in finite time, or $T = \infty$, in which case the process escapes to infinity, namely $Z(t) \to \infty$ when $t \to \infty$ (Theorem 10).

### *2.1. Limits of the hitting probability*

Our first key results of the section (Theorems 4 and 6) concern the probability of the process hitting the origin. We wish to show that $\lim_{\|(u,v)\|\to\infty} \mathbb{P}_{(u,v)}[T = \infty] = 1$. We shall prove this with the aid of Lemma 1 and Proposition 3.

For ease of notation, let us define $\tau_1^\xi := \inf\{t : Z_1(t \wedge T) \leq \xi\}$ and $\tau_2^\xi := \inf\{t : Z_2(t \wedge T) \leq \xi\}$. Further, let $X_1(t) := u + W_1(t) + \mu_1 t$ and let $X_2(t) := v + W_2(t) + \mu_2 t$.

Suppose $Z(t)$ is a one-dimensional reflected Brownian motion. The analysis of $Z(t)$ is converted to that of one-dimensional Brownian motion with a drift by the Skorokhod map. However, in the case of obliquely reflected Brownian motion in a quadrant, this method does not generally work due to the presence of $l_1(t)$ and $l_2(t)$. However, on the event $\{\tau_1^\xi = \infty\}$, note that $l_1(t) = 0$, and the previously reflected Brownian motion becomes an obliquely reflected Brownian motion in a half-plane. This allows one-dimensional techniques to be applied to the present problem. These considerations motivate us to consider the event $\{\tau_1^\xi\}$ below.

**Lemma 1.** *For $u > \xi > 0$, we have*

$$\mathbb{P}_{(u,v)}[\tau_1^\xi = \infty] = \mathbb{P}_{(u,v)}\left[X_1(t \wedge T) - r_2 \sup_{0\leq s\leq t\wedge T} (-X_2(s))^+ > \xi \text{ for every } t \geq 0\right], \tag{5}$$

*where $x^+$ equals $x$ if $x > 0$ and is $0$ otherwise. Hence,*

$$\mathbb{P}_{(u,v)}[\tau_1^\xi = \infty] \geq \mathbb{P}_{(u,v)}\left[X_1(t) - r_2 \sup_{0\leq s\leq t} (-X_2(s))^+ > \xi \text{ for every } t \geq 0\right]. \tag{6}$$

*A symmetrical result holds for $v > \xi > 0$ and $\mathbb{P}_{(u,v)}[\tau_2^\xi = \infty]$.*

**Proof.** On the event $\{\tau_1^\xi = \infty\}$, for every $t \geq 0$, we have $l_1(t) = 0$, $\mathbb{P}_{(u,v)}$-a.s. Then

$$Z_1(t \wedge T) = X_1(t \wedge T) - r_2 l_2(t \wedge T),$$
$$Z_2(t \wedge T) = X_2(t \wedge T) + l_2(t \wedge T).$$

Note that $l_2(t \wedge T)$ increases only when $Z_2(t \wedge T) = 0$. By uniqueness of the Skorokhod map (see e.g. [38] and references therein)

$$l_2(t \wedge T) = \sup_{0\leq s\leq t} (-X_2(s \wedge T))^+ = \sup_{0\leq s\leq t\wedge T} (-X_2(s))^+.$$

Thus

$$Z_1(t \wedge T) = X_1(t \wedge T) - r_2 \sup_{0 \le s \le t \wedge T} (-X_2(s))^+.$$

We may then write

$$\begin{aligned} &\{\tau_1^\xi = \infty\} = \{Z_1(t \wedge T) > \xi \text{ for every } t \ge 0\} \\ &= \{Z_1(t \wedge T) > \xi \text{ for every } t \ge 0 \text{ and } l_1(T) = 0\} \\ &= \left\{X_1(t \wedge T) - r_2 \sup_{0 \le s \le t \wedge T} (-X_2(s))^+ > \xi \text{ for every } t \ge 0 \text{ and } l_1(T) = 0\right\}, \end{aligned} \tag{7}$$

$\mathbb{P}_{(u,v)}$-a.s. We now wish to show that

$$\mathbb{P}_{(u,v)}\left[X_1(t \wedge T) - r_2 \sup_{0 \le s \le t \wedge T} (-X_2(s))^+ > \xi \text{ for every } t \ge 0 \text{ and } l_1(T) > 0\right] = 0. \tag{8}$$

Note that there is a set $N$ such that $\mathbb{P}_{(u,v)}(N) = 1$ and for every $\omega \in N$, we have

$$Z_1(t \wedge T) = X_1(t \wedge T) + l_1(t \wedge T) - r_2 l_2(t \wedge T) \ge 0, \tag{9}$$

$$Z_2(t \wedge T) = X_2(t \wedge T) - r_1 l_1(t \wedge T) + l_2(t \wedge T) \ge 0, \tag{10}$$

$$l_1(t \wedge T) \text{ increases only when } Z_1(t \wedge T) = 0, \tag{11}$$

$$l_2(t \wedge T) \text{ increases only when } Z_2(t \wedge T) = 0. \tag{12}$$

Let $\omega \in N$. We claim that the following statements

(a) $X_1(t \wedge T) - r_2 \sup_{0 \le s \le t \wedge T} (-X_2(s))^+ > \xi$ for every $t \ge 0$;
(b) $l_1(T) > 0$,

cannot hold simultaneously. The proof is by contradiction. For sake of contradiction, assume that statements a) and b) hold simultaneously. By (10), (12), and the uniqueness of Skorokhod map, we have

$$\begin{aligned} &l_2(t \wedge T) = \sup_{0 \le s \le t} (r_1 l_1(s \wedge T) - X_2(s \wedge T))^+ \\ &\le \sup_{0 \le s \le t} (r_1 l_1(s \wedge T))^+ + \sup_{0 \le s \le t} (-X_2(s \wedge T))^+ \\ &= r_1 l_1(t \wedge T) + \sup_{0 \le s \le t \wedge T} (-X_2(s))^+. \end{aligned}$$

Let $\eta := \inf\{t : l_1(t \wedge T) \ge \xi/(2r_1 r_2)\}$. Then for every $t \ge 0$,

$$\begin{aligned} &Z_1(t \wedge \eta \wedge T) = X_1(t \wedge \eta \wedge T) + l_1(t \wedge \eta \wedge T) - r_2 l_2(t \wedge \eta \wedge T) \\ &\ge X_1(t \wedge \eta \wedge T) - r_2 l_2(t \wedge \eta \wedge T) \\ &\ge X_1(t \wedge \eta \wedge T) - r_2 \sup_{0 \le s \le t \wedge \eta \wedge T} (-X_2(s))^+ - r_1 r_2 l_1(t \wedge \eta \wedge T) \\ &> \xi - \xi/2 = \xi/2, \end{aligned}$$

where in the last inequality we have invoked statement a). Since $l_1(t \wedge T)$ increases only when $Z_1(t \wedge T) = 0$, we have

$$l_1(t \wedge \eta \wedge T) = 0 \text{ for every } t \ge 0,$$

which contradicts statement b) and the definition of $\eta$. Therefore, by contradiction, (8) holds. Combining (7) and (8), (5) follows. Note that (6) follows directly from (5). □

**Remark 2.** To estimate the probability of the event

$$\{X_1(t) - r_2 \sup_{0\le s\le t}(-X_2(s))^+ > \xi \text{ for every } t \ge 0\},$$

we note that the above event contains the intersection of the event $\{X_1(t) > \xi + c$ for every t$\}$ and the event $\{\sup_{0\le s\le t}(-X_2(s))^+ < c/r_2$ for every t$\}$ for every positive $c$, both of which correspond to the first hitting problems of one-dimensional Brownian motion with a drift. We will use the idea repeatedly in the proofs of Theorem 4 and Lemma 8.

We now turn to Proposition 3, which is a reformulation of the formula 1.2.4(1) on p. 252 of [5].

**Proposition 3.** *Let $B(t)$ be a one dimensional Brownian motion started from the origin under $\mathbb{P}$. For $\mu > 0$ and $x > 0$, we have*

$$\mathbb{P}(B(t) + \mu t > -x \textit{ for every } t \ge 0) = 1 - e^{-2x\mu}.$$

With Lemma 1 and Proposition 3 in hand, we state Theorem 4.

**Theorem 4.** *When the starting point tends to infinity, the probability that the process does not hit the origin tends to one. Namely,*

$$\lim_{\|(u,v)\|\to\infty} \mathbb{P}_{(u,v)}[T = \infty] = 1.$$

*Equivalently,*

$$\lim_{\|(u,v)\|\to\infty} \mathbb{P}_{(u,v)}[T < \infty] = 0.$$

**Proof.** Fix $\xi > 0$. For $\|(u, v)\|$ sufficiently large, we have $u > 2\xi$ or $v > 2\xi$. If $u > 2\xi$, by Lemma 1, we have

$$\begin{aligned}
&\mathbb{P}_{(u,v)}[T = \infty\ ] \ge \mathbb{P}_{(u,v)}[\tau_1^{\xi} = \infty] \\
&\ge \mathbb{P}_{(u,v)}\left[X_1(t) - r_2 \sup_{0\le s\le t}(-X_2(s))^+ > \xi \text{ for every } t \ge 0\right] \\
&\ge \mathbb{P}_{(u,v)}[X_1(t) > \xi + u/2 \text{ for every } t \ge 0 \text{ and } X_2(t) > -u/(2r_2) \text{ for every } t \ge 0] \\
&\ge \mathbb{P}_{(u,v)}[X_1(t) > \xi + u/2 \text{ for every } t \ge 0] \\
&\quad + \mathbb{P}_{(u,v)}[X_2(t) > -u/(2r_2) \text{ for every } t \ge 0] - 1 \\
&= \mathbb{P}_{(u,v)}[W_1(t) + \mu_1 t > -(u - 2\xi)/2 \text{ for every } t \ge 0] \\
&\quad + \mathbb{P}_{(u,v)}[W_2(t) + \mu_2 t > -u/(2r_2) - v \text{ for every } t \ge 0] - 1 \\
&= 1 - e^{-(u-2\xi)\mu_1} + 1 - e^{-(u/r_2+2v)\mu_2} - 1 \\
&= 1 - e^{-(u-2\xi)\mu_1} - e^{-(u/r_2+2v)\mu_2},
\end{aligned}$$

where the second to last equality invokes Proposition 3. Similarly, if $v > 2\xi$, we have

$$\mathbb{P}_{(u,v)}[T = \infty\ ] \ge 1 - e^{-(v-2\xi)\mu_2} - e^{-(v/r_1+2u)\mu_1}.$$

Hence,

$$\begin{aligned}
&\mathbb{P}_{(u,v)}[T = \infty] \\
&\ge \max\{(1 - e^{-(u-2\xi)\mu_1} - e^{-(u/r_2+2v)\mu_2})1_{\{u>2\xi\}}, (1 - e^{-(v-2\xi)\mu_2} - e^{-(v/r_1+2u)\mu_1})1_{\{v>2\xi\}}\}.
\end{aligned}$$

Letting $(u, v)$ tend to $\infty$, the desired result follows. □

We now turn to Proposition 5, which shall be needed to prove Theorem 6.

**Proposition 5.** *We have the following subset relationship*

$$\{u + W_1(t) + \mu_1 t < 0 \text{ and } v + W_2(t) + \mu_2 t < 0, \text{ for some } t \in \mathbb{R}_+\} \subset \{T < \infty\}.$$

**Proof.** We prove this claim by contradiction. For the sake of contradiction, let us fix $\omega \in \{u + W_1(t) + \mu_1 t < 0 \text{ and } v + W_2(t) + \mu_2 t < 0, \text{ for some } t \in \mathbb{R}_+\} \cap \{T = \infty\}$. Assuming $T = \infty$, the process can be written as

$$\begin{cases} Z_1(t) = u + W_1(t) + \mu_1 t + l_1(t) - r_2 l_2(t), \\ Z_2(t) = v + W_2(t) + \mu_2 t - r_1 l_1(t) + l_2(t). \end{cases}$$

Solving the linear system for $l_1$ and $l_2$, we obtain

$$\begin{cases} (r_1 r_2 - 1) l_1(t) = (u + W_1(t) + \mu_1 t - Z_1(t)) + r_2(v + W_2(t) + \mu_2 t - Z_2(t)), \\ (r_1 r_2 - 1) l_2(t) = r_1(u + W_1(t) + \mu_1 t - Z_1(t)) + (v + W_2(t) + \mu_2 t - Z_2(t)). \end{cases}$$

For all $t \in \mathbb{R}_+$ such that

$$u + W_1(t) + \mu_1 t < 0,$$

and

$$v + W_2(t) + \mu_2 t < 0,$$

we have $(r_1 r_2 - 1) l_1(t) < 0$ and $(r_1 r_2 - 1) l_2(t) < 0$, which is not possible since $l_1(t)$ and $l_2(t) \geqslant 0$ and we have assumed $(r_1 r_2 - 1) \geqslant 0$. A contradiction has been reached. □

Theorem 6 considers the behavior of the process when the starting point tends to the origin.

**Theorem 6.** *When the starting point tends to the origin, the probability that the process hits the origin in finite time tends towards one. That is,*

$$\lim_{(u,v)\to(0,0)} \mathbb{P}_{(u,v)}[T < \infty] = 1,$$

*or equivalently,*

$$\lim_{(u,v)\to(0,0)} \mathbb{P}_{(u,v)}[T = \infty] = 0.$$

**Proof.** By Proposition 5, we have that

$$\mathbb{P}_{(u,v)}[T < \infty] \geqslant \mathbb{P}[\exists t \in \mathbb{R}_+ : u + W_1(t) + \mu_1 t < 0 \text{ and } v + W_2(t) + \mu_2 t < 0].$$

By the properties of planar Brownian motion, we have

$$\mathbb{P}[\exists t \in \mathbb{R}_+ : W_1(t) + \mu_1 t < 0 \text{ and } W_2(t) + \mu_2 t < 0] = 1.$$

Let $(u_n, v_n) \in \mathbb{R}_+^2$ be a sequence of points such that $(u_n, v_n) \to (0, 0)$. Note that

$$\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\exists t \in \mathbb{R}_+ : u_n + W_1(t) + \mu_1 t < 0 \text{ and } v_n + W_2(t) + \mu_2 t < 0\}$$
$$\supset \{\exists t \in \mathbb{R}_+ : W_1(t) + \mu_1 t < 0 \text{ and } W_2(t) + \mu_2 t < 0\}.$$

Applying Fatou's Lemma yields

$$\liminf_{n\to\infty} \mathbb{P}\left[\exists t \in \mathbb{R}_+ : u_n + W_1(t) + \mu_1 t < 0 \text{ and } v_n + W_2(t) + \mu_2 t < 0\right] \geqslant 1.$$

We may therefore conclude that

$$\mathbb{P}_{(u_n,v_n)}\left[T < \infty\right] \underset{n\to\infty}{\longrightarrow} 1,$$

and the desired result follows. □

## 2.2. *Complementarity of absorption and escape*

We now turn to Theorem 10, which states that the process has only two possible behaviors: either $T < \infty$, or $T = \infty$, in which case $Z(t) \to \infty$ when $t \to \infty$. The result first requires the proofs of three auxiliary statements which we give below.

**Proposition 7.** *Suppose $B(t)$ is a one dimensional Brownian motion starting from the origin under the measure $\mathbb{P}$. Let $a, b$ be two positive numbers. Then*

$$\mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0) > 0.$$

**Proof.** Let $\lambda = \ln 2/(2b) + 1$. Note that $1 - 2\,e^{-2\lambda b} > 0$. Then

$$\begin{aligned} &\mathbb{P}(-\lambda - bt < B(t) < \lambda + bt \text{ for every } t \geq 0)\\ \geq\ &\mathbb{P}(B(t) > -\lambda - bt \text{ for every } t \geq 0) + \mathbb{P}(B(t) < \lambda + bt \text{ for every } t \geq 0) - 1\\ =\ &2(1 - e^{-2\lambda b}) - 1 = 1 - 2\,e^{-2\lambda b} > 0. \end{aligned}$$

Let $H_a := \inf\{t : |B(t)| = a\}$. By standard exit time properties of Brownian motion, $\mathbb{P}(H_a > \lambda/b + 1) > 0$. Then

$$\begin{aligned} &\mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0)\\ =\ &\mathbb{P}(H_a > \lambda/b + 1)\,\mathbb{P}(-a - bt < B(t) < a + bt \text{ for every } t \geq 0 \,|H_a > \lambda/b + 1). \end{aligned}$$

By the strong Markov property of Brownian motion,

$$\begin{aligned} &\mathbb{P}(-a - bt < B(t) < a + bt, \forall t \,|H_a > \lambda/b + 1)\\ =\ &\mathbb{P}(-a - b(t + H_a) < B(t + H_a) < a + b(t + H_a), \forall t \,|H_a > \lambda/b + 1)\\ =\ &\mathbb{P}(\,-a - b(t + H_a) - B(H_a) < B(t + H_a) - B(H_a) < a\\ &\quad +b(t + H_a) - B(H_a), \forall t \,|H_a > \lambda/b + 1)\\ \geq\ &\mathbb{P}(-\lambda - bt < B(t + H_a) - B(H_a) < \lambda + bt, \forall t \,|H_a > \lambda/b + 1)\\ =\ &\mathbb{P}(-\lambda - bt < B(t) < \lambda + bt, \forall t\,)\\ >\ &0, \end{aligned}$$

from which the desired result follows. □

We now turn to Lemma 8.

**Lemma 8.** *For a positive number $\alpha$,*

$$\inf_{u\geq\alpha} \mathbb{P}_{(u,0)}[\tau_1^0 = \infty] > 0, \tag{13}$$

$$\inf_{v\geq\alpha} \mathbb{P}_{(0,v)}[\tau_2^0 = \infty] > 0. \tag{14}$$

**Proof.** We need only prove (13), since the proof of (14) is completely symmetric. Let us consider $\xi < \alpha$. By Lemma 1,

$$\begin{aligned}
&\mathbb{P}_{(u,0)}[\tau_1^0 = \infty] \geq \mathbb{P}_{(u,0)}[\tau_1^\xi = \infty] \\
&\geq \mathbb{P}_{(u,0)}\left[X_1(t) - r_2 \sup_{0\leq s\leq t}(-X_2(s))^+ > \xi \text{ for every } t \geq 0\right] \\
&= \mathbb{P}_{(u,0)}\left[u + W_1(t) + \mu_1 t - r_2 \sup_{0\leq s\leq t}(-W_2(s) - \mu_2 t)^+ > \xi \text{ for every } t \geq 0\right] \\
&\geq \mathbb{P}_{(u,0)}[W_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0 \\
&\qquad \text{and } W_2(t) + \mu_2 t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0]. \qquad (15)
\end{aligned}$$

Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions starting from 0 under $\mathbb{P}_{(u,0)}$. Then, under $\mathbb{P}_{(u,0)}$, the process $(W_1(t), W_2(t))$ has the same law as $(B_1(t), \rho B_1(t) + \sqrt{1-\rho^2}B_2(t))$. We now show that (13) holds in three separate cases: $\rho = 0$, $0 < \rho < 1$ and $-1 < \rho < 0$.

*Case I:* $\rho = 0$. If $\rho = 0$, then $W_1(t)$ and $W_2(t)$ are two independent Brownian motions. Then

$$\begin{aligned}
(15) &= \mathbb{P}_{(u,0)}[W_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0] \\
&\quad \times \mathbb{P}_{(u,0)}[W_2(t) + \mu_2 t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0] \\
&= \left(1 - e^{-(u-\xi)\mu_1}\right) \cdot \left(1 - e^{-(u-\xi)\mu_2/r_2}\right),
\end{aligned}$$

where the last equality invokes Proposition 3. Taking infimums yields

$$\inf_{u\geq\alpha} \mathbb{P}_{(u,0)}[\tau_1^0 = \infty] \geq \left(1 - e^{-(\alpha-\xi)\mu_1}\right) \cdot \left(1 - e^{-(\alpha-\xi)\mu_2/r_2}\right) > 0.$$

*Case II:* $0 < \rho < 1$. If $0 < \rho < 1$, then

$$\begin{aligned}
(15) &= \mathbb{P}_{(u,0)}[B_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0 \\
&\qquad \text{and } \rho B_1(t) + \sqrt{1-\rho^2}B_2(t) + \mu_2 t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0] \\
&\geq \mathbb{P}_{(u,0)}[B_1(t) + (\mu_1 \wedge \mu_2)t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0 \\
&\qquad \text{and } \sqrt{1-\rho^2}B_2(t) + (1-\rho)\mu_2 t > -(1-\rho)(u-\xi)/(2r_2) \text{ for every } t \geq 0].
\end{aligned}$$

Using the same argument in the case for $\rho = 0$, (13) follows.

*Case III:* $-1 < \rho < 0$. If $-1 < \rho < 0$, then for $u \geq \alpha$

$$\begin{aligned}
(15) &= \mathbb{P}_{(u,0)}[B_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0 \\
&\qquad \text{and } \rho B_1(t) + \sqrt{1-\rho^2}B_2(t) + \mu_2 t > -(u-\xi)/(2r_2) \text{ for every } t \geq 0] \\
&\geq \mathbb{P}_{(u,0)}[B_1(t) + \mu_1 t > -(u-\xi)/2 \text{ for every } t \geq 0, \\
&\qquad \rho B_1(t) - \rho(\mu_1 \wedge \mu_2)t > -|\rho|(u-\xi)/(2r_2) \text{ for every } t \geq 0 \\
&\quad \text{and } \sqrt{1-\rho^2}B_2(t) + \mu_2 + \rho(\mu_1 \wedge \mu_2)t > -(1-|\rho|)(u-\xi)/(2r_2) \text{ for every } t \geq 0] \\
&\geq \mathbb{P}_{(u,0)}[-(u-\xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (u-\xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t \\
&\quad \text{and } \sqrt{1-\rho^2}B_2(t) + \mu_2 + \rho(\mu_1 \wedge \mu_2)t > -(1-|\rho|)(u-\xi)/(2r_2), \forall t] \\
&= \mathbb{P}_{(u,0)}[-(u-\xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (u-\xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t] \\
&\quad \times \mathbb{P}_{(u,0)}[\sqrt{1-\rho^2}B_2(t) + \mu_2 + \rho(\mu_1 \wedge \mu_2)t > -(1-|\rho|)(u-\xi)/(2r_2), \forall t] \\
&\geq \mathbb{P}_{(u,0)}[-(\alpha-\xi)/(2r_2) - (\mu_1 \wedge \mu_2)t < B_1(t) < (\alpha-\xi)/(2r_2) + (\mu_1 \wedge \mu_2)t, \forall t] \\
&\quad \times \mathbb{P}_{(u,0)}[\sqrt{1-\rho^2}B_2(t) + \mu_2 + \rho(\mu_1 \wedge \mu_2)t > -(1-|\rho|)(\alpha-\xi)/(2r_2), \forall t].
\end{aligned}$$

Taking infimums and invoking Proposition 7, (13) follows. This concludes the proof. $\square$

Let us define $T_r := \inf\{t \geq 0 : \|Z(t \wedge T)\| \leq r\}$. We proceed with Lemma 9 below.

**Lemma 9.** *For fixed $n$, on the event $\{T_{1/n} = \infty\}$, we have ($\mathbb{P}_{(u,v)}$-a.s.) that*

$$\lim_{t\to\infty} Z(t) = \infty.$$

*That is,*

$$\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0. \tag{16}$$

**Proof.** We will first show that (16) holds when $v = 0$. Then (16) will follow immediately in the case $u = 0$. We shall conclude by showing that (16) holds when $u \neq 0$ and $v \neq 0$.

*Case I:* $v = 0$. When $v = 0$, let

$$K := \sup_{u\geq 0} \mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right].$$

For $u \leq 1/n$,

$$\mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0.$$

Then

$$K = \sup_{u\geq 1/n} \mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]. \tag{17}$$

We now define a stopping time

$$\eta_1^0 := \begin{cases} \inf\{t \geq \tau_1^0 : Z_2(t) = 0\}, & \tau_1^0 < \infty, \\ \infty, & \tau_1^0 = \infty. \end{cases}$$

By Lemma 8,

$$\inf_{u\geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 = \infty] \geq \inf_{u\geq 1/n} \mathbb{P}_{(u,0)}[\tau_1^0 = \infty] > 0,$$

and hence,

$$\sup_{u\geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 < \infty] < 1. \tag{18}$$

Note that

$$\begin{aligned} &\mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ &= \mathbb{P}_{(u,0)}\left[\tau_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ &\quad + \mathbb{P}_{(u,0)}\left[\tau_1^0 < \infty, \eta_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\ &\quad + \mathbb{P}_{(u,0)}\left[\eta_1^0 < \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]. \end{aligned} \tag{19}$$

On the event $\{\tau_1^0 = \infty\}$, for all $t \geq 0$, $T = \infty$ and $l_1(t) = 0$. Then

$$Z_2(t) = X_2(t) + l_2(t) \geq X_2(t) = W_2(t) + \mu_2 t \to \infty,$$

$\mathbb{P}_{(u,0)}$-a.s., by the law of the iterated logarithm for Brownian motion. Hence, the first term on the right-hand side of (19) is 0. We now consider the second term on the right-hand side of (19). On the event $\{\tau_1^0 < \infty\}$, let us define $\tilde{\eta}_1^0 := \inf\{t \geq 0 : Z_2(t+\tau_1^0) = 0\}$ and $\tilde{T}_{1/n} := \inf\{t \geq 0 : \|Z(t+\tau_1^0)\| \leq 1/n\}$. By the strong Markov property, we have

$$
\begin{aligned}
&\mathbb{P}_{(u,0)}\left[\tau_1^0 < \infty, \eta_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\
&= \mathbb{P}_{(u,0)}\left[\tau_1^0 < \infty, \inf_{0\leq s\leq \tau_1^0} \|Z(s)\| > \frac{1}{n}, \tilde{\eta}_1^0 = \infty, \liminf_{t\to\infty} Z(t+\tau_1^0) < \infty, \tilde{T}_{\frac{1}{n}} = \infty\right] \\
&= \mathbb{E}_{(u,0)}\left[\mathbb{1}_{\{\tau_1^0<\infty, \inf_{0\leq s\leq\tau_1^0}\|Z(s)\|>1/n\}}\, \mathbb{P}_{Z(\tau_1^0)}\left[\eta_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]\right] \\
&= 0.
\end{aligned}
$$

By the same argument used to show that the first term on the right-hand side of (19) is 0, for $v > 0$,

$$
\mathbb{P}_{(0,v)}\left[\eta_1^0 = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0.
$$

This proves that the second term on the right-hand side of (19) is also 0. We now consider the third term on the right-hand side of (19). On the event $\{\eta_1^0 < \infty\}$, let $\hat{T}_{1/n} := \inf\{t \geq 0 : Z(t+\eta_1^0) \leq 1/n\}$. By the strong Markov property,

$$
\begin{aligned}
&\mathbb{P}_{(u,0)}\left[\eta_1^0 < \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\
&= \mathbb{P}_{(u,0)}\left[\eta_1^0 < \infty, \inf_{0\leq s\leq \eta_1^0} \|Z(s)\| > \frac{1}{n}, \liminf_{t\to\infty} Z(t+\eta_1^0) < \infty, \hat{T}_{\frac{1}{n}} = \infty\right] \\
&= \mathbb{E}_{(u,0)}\left[\mathbb{1}_{\{\eta_1^0<\infty, \inf_{0\leq s\leq\eta_1^0}\|Z(s)\|>1/n\}}\, \mathbb{P}_{Z(\eta_1^0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]\right] \\
&\leq K \cdot \mathbb{E}_{(u,0)}\left[\mathbb{1}_{\{\eta_1^0<\infty, \inf_{0\leq s\leq\eta_1^0}\|Z(s)\|>1/n\}}\right] \\
&\leq K \cdot \mathbb{P}_{(u,0)}[\eta_1^0 < \infty].
\end{aligned}
$$

Combining (19) and the above estimates yields

$$
\mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \leq K \cdot \mathbb{P}_{(u,0)}[\eta_1^0 < \infty].
$$

Taking supremums and invoking (17), we obtain

$$
K = \sup_{u\geq 1/n} \mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \leq K \cdot \sup_{u\geq 1/n} \mathbb{P}_{(u,0)}[\eta_1^0 < \infty].
$$

Together with (18), we have $K = 0$. Hence, for every $u \geq 0$,

$$
\mathbb{P}_{(u,0)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0. \tag{20}
$$

Similarly, for every $v \geq 0$,

$$
\mathbb{P}_{(0,v)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0. \tag{21}
$$

*Case II: $u \neq 0$ and $v \neq 0$.* For the case when $u \neq 0$ and $v \neq 0$, let $\tau := \inf\{t \geq 0 : Z_1(t) = 0 \text{ or } Z_2(t) = 0\}$. Then

$$\begin{aligned}
&\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\
&= \mathbb{P}_{(u,v)}\left[\tau = \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\
&\quad + \mathbb{P}_{(u,v)}\left[\tau < \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]. \qquad (22)
\end{aligned}$$

On the event $\{\tau = \infty\}$, $T = \infty$ and, for every $t \geq 0$, $l_1(t) = l_2(t) = 0$. Then, as $t \to \infty$,

$$Z_1(t) = u + W_1(t) + \mu_1 t \to \infty,$$

$\mathbb{P}_{(u,v)}$-a.s. Hence the first term on the right-hand side of (22) is 0. We now consider the second term on the right-hand side of (22). By the strong Markov property,

$$\begin{aligned}
&\mathbb{P}_{(u,v)}\left[\tau < \infty, \liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] \\
&\leq \mathbb{E}_{(u,v)}\left[\mathbb{1}_{\{\tau<\infty\}}\, \mathbb{P}_{Z(\tau)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right]\right] \\
&= 0,
\end{aligned}$$

where (20) and (21) have been invoked in the last equality. Hence the second term on the right-hand side of (22) is also 0. Thus for $u \neq 0$ and $v \neq 0$,

$$\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty, T_{\frac{1}{n}} = \infty\right] = 0.$$

The proof is now complete. □

With the above results in hand, we now state Theorem 10.

**Theorem 10.** *On the event $\{T = \infty\}$, $\mathbb{P}_{(u,v)}$-a.s. the process $Z(t)$ tends to infinity when $t \to \infty$, namely*

$$\mathbb{P}_{(u,v)}\left[\lim_{t\to\infty} Z(t) = \infty \,\middle|\, T = \infty\right] = 1.$$

*Equivalently,*

$$\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\ T = \infty\right] = 0.$$

**Proof.** We have from Lemma 9 that for every $n \in \mathbb{N}_+^*$

$$\begin{aligned}
&\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\ T = \infty\right] \\
&= \mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\ T_{\frac{1}{n}} < \infty,\ T = \infty\right].
\end{aligned}$$

Applying the strong Markov property yields

$$\begin{aligned}
&\mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\ T_{\frac{1}{n}} < \infty,\ T = \infty\right] \\
&= \mathbb{E}_{(u,v)}\left[\mathbb{1}_{\{T_{1/n}<\infty\}}\, \mathbb{P}_{Z(T_{1/n})}\left[\liminf_{t\to\infty} Z(t) < \infty,\ T = \infty\right]\right]
\end{aligned}$$

$$
\leq \sup_{\|(u,v)\|=1/n} \mathbb{P}_{(u,v)}\left[\liminf_{t\to\infty} Z(t) < \infty,\ T = \infty\right]
$$
$$
\leq \sup_{\|(u,v)\|=1/n} \mathbb{P}_{(u,v)}[T = \infty].
$$

Applying Theorem 6 and letting $n \to \infty$, the desired result follows. □

## 3. Partial differential equation and functional equation

We now turn to the study of the escape probability $\mathbb{P}_{(u,v)}[T = \infty]$. We begin with Proposition 11, which provides partial differential equations characterizing the escape probability. We then proceed with Proposition 12, which gives a functional equation satisfied by the Laplace transform of the escape probability. Note that there is no particular difficulty in defining the process starting from the edge (except the origin).

Let us define the infinitesimal generator of the process inside the quarter plane as

$$
\mathcal{G} f(u,v) := \lim_{t\to 0} \frac{1}{t}\mathbb{E}_{(u,v)}[f(Z(t\wedge T)) - f(u,v)],
$$

where $f$ must be a bounded function in the quadrant to ensure that the above limit exists and is uniform. For $f$ twice differentiable, the infinitesimal generator inside the quadrant is

$$
\mathcal{G} f = \frac{1}{2}\left(\frac{\partial^2 f}{\partial z_1^2} + \frac{\partial^2 f}{\partial z_2^2} + 2\rho\frac{\partial^2 f}{\partial z_1 \partial z_2}\right) + \mu_1 \frac{\partial f}{\partial z_1} + \mu_2 \frac{\partial f}{\partial z_2}.
$$

This leads us to Proposition 11.

**Proposition 11** (*Partial Differential Equation*)**.** *The absorption probability*

$$
f(u,v) = \mathbb{P}_{(u,v)}[T < \infty],
$$

*is the only function which is both (i) bounded and continuous in the quarter plane and on its boundary and (ii) continuously differentiable in the quarter plane and on its boundary (except perhaps at the corner), and which satisfies the partial differential equation*

$$
\mathcal{G} f(u,v) = 0, \quad \forall (u,v) \in \mathbb{R}_+^2,
$$

*with oblique Neumann boundary conditions*

$$
\begin{cases} \partial_{r_1} f(0,v) := \frac{\partial f}{\partial u}(0,v) - r_1 \frac{\partial f}{\partial v}(0,v) = 0 & \forall v > 0, \\ \partial_{r_2} f(u,0) := -r_2 \frac{\partial f}{\partial u}(u,0) + \frac{\partial f}{\partial v}(u,0) = 0 & \forall u > 0, \end{cases} \tag{23}
$$

*and with limit values*

$$
\begin{cases} f(0,0) = 1, \\ \lim_{(u,v)\to\infty} f(u,v) = 0. \end{cases}
$$

*The same result holds for the escape probability*

$$
g(u,v) = 1 - f(u,v) = \mathbb{P}_{(u,v)}[T = \infty]
$$

*but with the following limit values*

$$
\begin{cases} g(0,0) = 0, \\ \lim_{\|(u,v)\|\to\infty} f(u,v) = 1. \end{cases}
$$

**Proof.** This proof is inspired by Foddy [16, p. 86–89]. We assume that $f$ satisfies the hypotheses of the Proposition. Applying Dynkin's formula, we obtain

$$\begin{aligned}\mathbb{E}_{(u,v)}[f(Z(t\wedge T))] &= f(u,v) + \mathbb{E}_{(u,v)}\int_0^{t\wedge T} \mathcal{G}f(Z(s))\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{2}\mathbb{E}_{(u,v)}\int_0^{t\wedge T} \partial_{r_i} f(Z(s))\,\mathrm{d}l_i(s) \\ &= f(u,v).\end{aligned}$$

But,

$$\begin{aligned}\mathbb{E}_{(u,v)}[f(Z(t\wedge T))] &= \mathbb{E}_{(u,v)}[f(Z(t\wedge T))\mathbb{1}_{T\leqslant t}] + \mathbb{E}_{(u,v)}[f(Z(t\wedge T))\mathbb{1}_{T>t}] \\ &= f(0,0)\mathbb{P}_{(u,v)}[T\leqslant t] + \mathbb{E}_{(u,v)}[f(Z(t))\mathbb{1}_{T>t}] \\ &\underset{t\to\infty}{\longrightarrow} \mathbb{P}_{(u,v)}[T<\infty] + \lim_{t\to\infty}\mathbb{E}_{(u,v)}[f(Z(t))\mathbb{1}_{T>t}] \\ &= \mathbb{P}_{(u,v)}[T<\infty].\end{aligned}$$

Note that $\lim_{|z|\to\infty} f(z) = 0$ and that for $T > t$, $Z(t) \underset{t\to\infty}{\to} \infty$ a.s. By dominated convergence and by Theorem 10,

$$\lim_{t\to\infty}\mathbb{E}_{(u,v)}[f(Z(t))\mathbb{1}_{T>t}] = \mathbb{E}_{(u,v)}[\lim_{t\to\infty} f(Z(t))\mathbb{1}_{T=\infty}] = 0.$$

We may thus conclude that

$$f(u,v) = \mathbb{P}_{(u,v)}[T<\infty].$$

Conversely, denote $f(u,v) := \mathbb{P}_{(u,v)}[T<\infty]$. The function $f$ is bounded. By the Markov property, we have that

$$\mathbb{E}_{(u,v)}[f(Z(t\wedge T))] = f(u,v).$$

Since

$$\mathcal{G}f(u,v) = \lim_{t\to 0}\frac{1}{t}(\mathbb{E}_{(u,v)}[f(Z(t\wedge T))] - f(u,v)) = 0,$$

we may conclude that $\mathcal{G}f=0$ on the quarter plane. The continuity and differentiability properties of $f$ are immediate from Andres [2, Thm 2.2 and Cor 2.4]. One may also refer to [35] which establishes these properties in a greater generality. The Neumann boundary condition is satisfied by applying [2, Cor 3.3]. The desired limit values at 0 and at infinity are obtained by invoking Theorem 4 and Theorem 6. The result for $g = 1 - f$ is straightforward, and this completes the proof. □

In preparation for Proposition 12, let us define the Laplace transform of the escape probability starting from $(u,v)$ as

$$\psi(x,y) := \int_0^\infty\int_0^\infty e^{-xu-yv}\mathbb{P}_{(u,v)}[T=\infty]\,\mathrm{d}u\mathrm{d}v.$$

Further, let

$$\psi_1(x) := \int_0^\infty e^{-xu}\mathbb{P}_{(u,0)}[T=\infty]\,\mathrm{d}u \quad\text{and}\quad \psi_2(y) := \int_0^\infty e^{-yv}\mathbb{P}_{(0,v)}[T=\infty]\,\mathrm{d}v. \tag{24}$$

We also define the kernel

$$K(x,y) := \frac{1}{2}(x^2+y^2+2\rho xy) + \mu_1 x + \mu_2 y, \tag{25}$$

and let

$$k_1(x,y) := \frac{1}{2}(r_2 x + y) + \rho x + \mu_2, \quad k_2(x,y) := \frac{1}{2}(x + r_1 y) + \rho y + \mu_1. \tag{26}$$

We now provide a functional equation satisfied by the Laplace transform of the escape probability.

**Proposition 12** (*Functional Equation*)**.** *For $(x, y) \in \mathbb{C}^2$ such that $\Re x > 0$ and $\Re y > 0$ we have*

$$K(x,y)\psi(x,y) = k_1(x,y)\psi_1(x) + k_2(x,y)\psi_2(y). \tag{27}$$

This functional equation is very similar to that obtained in [17, (32)] to compute an escape probability along one of the axes.

**Proof.** Recall the partial differential equation in Proposition 11 with the oblique Neumann boundary condition and limit values satisfied by $g(u,v) := \mathbb{P}_{(u,v)}[T = \infty]$. Employing integration by parts yields

$$\begin{aligned}
0 &= \int_0^\infty \int_0^\infty e^{-xz_1-yz_2} \mathcal{G} g(z_1,z_2)\,\mathrm{d}z_1\mathrm{d}z_2 \\
0 &= \int_0^\infty \frac{1}{2} e^{-yz_2} \left( -\frac{\partial g}{\partial z_1}(0,z_2) + x \int_0^\infty e^{-xz_1} \frac{\partial g}{\partial z_1}(z_1,z_2)\,\mathrm{d}z_1 \right) \mathrm{d}z_2 \\
&+ \int_0^\infty \frac{1}{2} e^{-xz_1} \left( -\frac{\partial g}{\partial z_2}(z_1,0) + y \int_0^\infty e^{-yz_2} \frac{\partial g}{\partial z_2}(z_1,z_2)\,\mathrm{d}z_2 \right) \mathrm{d}z_1 \\
&+ \int_0^\infty \rho e^{-xz_1} \left( -\frac{\partial g}{\partial z_1}(z_1,0) + y \int_0^\infty e^{-yz_2} \frac{\partial g}{\partial z_1}(z_1,z_2)\,\mathrm{d}z_2 \right) \mathrm{d}z_1 \\
&+ \int_0^\infty \mu_1 e^{-yz_2} \left( -g(0,z_2) + x \int_0^\infty e^{-xz_1} g(z_1,z_2)\,\mathrm{d}z_1 \right) \mathrm{d}z_2 \\
&+ \int_0^\infty \mu_2 e^{-xz_1} \left( -g(z_1,0) + y \int_0^\infty e^{-yz_2} g(z_1,z_2)\,\mathrm{d}z_2 \right) \mathrm{d}z_1 \\
0 &= -\frac{1}{2} r_1 \int_0^\infty e^{-yz_2} \frac{\partial g}{\partial z_2}(0,z_2)\,\mathrm{d}z_2 + \frac{x}{2} \int_0^\infty e^{-yz_2} \left( -g(0,z_2) + x \int_0^\infty e^{-xz_1} g(z_1,z_2)\,\mathrm{d}z_1 \right) \mathrm{d}z_2 \\
&- \frac{1}{2} r_2 \int_0^\infty e^{-xz_1} \frac{\partial g}{\partial z_1}(z_1,0)\,\mathrm{d}z_1 + \frac{y}{2} \int_0^\infty e^{-xz_1} \left( -g(z_1,0) + y \int_0^\infty e^{-yz_2} g(z_1,z_2)\,\mathrm{d}z_2 \right) \mathrm{d}z_1 \\
&- \rho \int_0^\infty e^{-xz_1} \frac{\partial g}{\partial z_1}(z_1,0)\,\mathrm{d}z_1 + \rho y \int_0^\infty e^{-yz_2} \left( -g(0,z_2) + x \int_0^\infty e^{-xz_1} g(z_1,z_2)\,\mathrm{d}z_1 \right) \mathrm{d}z_2 \\
&- \mu_1 \int_0^\infty e^{-yz_2} g(0,z_2)\,\mathrm{d}z_2 + \mu_1 x \int_0^\infty \int_0^\infty e^{-xz_1-yz_2} g(z_1,z_2)\,\mathrm{d}z_1\mathrm{d}z_2 \\
&- \mu_2 \int_0^\infty e^{-xz_1} g(z_1,0)\,\mathrm{d}z_1 + \mu_2 y \int_0^\infty \int_0^\infty e^{-xz_1-yz_2} g(z_1,z_2)\,\mathrm{d}z_1\mathrm{d}z_2 \\
0 &= \left( \frac{1}{2}(x^2 + y^2 + 2\rho xy) + \mu_1 x + \mu_2 y \right) \int_0^\infty \int_0^\infty e^{-xz_1-yz_2} g(z_1,z_2)\,\mathrm{d}z_1\mathrm{d}z_2 \\
&- \left( \frac{1}{2}(r_2 x + y) + \rho x + \mu_2 \right) \int_0^\infty e^{-xz_1} g(z_1,0)\,\mathrm{d}z_1 \\
&- \left( \frac{1}{2}(x + r_1 y) + \rho y + \mu_1 \right) \int_0^\infty e^{-yz_2} g(0,z_2)\,\mathrm{d}z_2 \\
0 &= K(x,y)\psi(x,y) - k_1(x,y)\psi_1(x) - k_2(x,y)\psi_2(y).
\end{aligned}$$

This concludes the proof. □

## 4. Kernel and asymptotics

We begin by studying some properties of the kernel $K$ as defined in (25). Note that this kernel is similar to that in [21] except that in the present paper the drift is positive. We consider the functions $X$ and $Y$ satisfying

$$K(X(y), y) = 0 \quad \text{and} \quad K(x, Y(x)) = 0.$$

The branches are given by

$$\begin{cases} X^{\pm}(y) := -(\rho y + \mu_1) \pm \sqrt{y^2(\rho^2 - 1) + 2y(\mu_1\rho - \mu_2) + \mu_1^2}, \\ Y^{\pm}(x) := -(\rho x + \mu_2) \pm \sqrt{x^2(\rho^2 - 1) + 2x(\mu_2\rho - \mu_1) + \mu_2^2}, \end{cases} \tag{28}$$

and the branch points of $X$ and $Y$ (which are roots of the polynomials in the square roots of (28)) are given, respectively, by

$$\begin{cases} y^{\pm} := \dfrac{\mu_1\rho - \mu_2 \pm \sqrt{(\mu_1\rho - \mu_2)^2 + \mu_1^2(1-\rho^2)}}{(1-\rho^2)}, \\ x^{\pm} := \dfrac{\mu_2\rho - \mu_1 \pm \sqrt{(\mu_2\rho - \mu_1)^2 + \mu_2^2(1-\rho^2)}}{(1-\rho^2)}. \end{cases} \tag{29}$$

By (3), we obtain that

$$y^+ = \mu_1 \frac{1 - \cos(\beta - \theta)}{\sin\beta \sin(\beta - \theta)}. \tag{30}$$

The functions $X^{\pm}$ and $Y^{\pm}$ are analytic on the cut planes $\mathbb{C} \setminus ((-\infty, y^-] \cup [y^+, \infty))$ and $\mathbb{C} \setminus ((-\infty, x^-] \cup [x^+, \infty))$, respectively. Fig. 4 below depicts the functions $Y^{\pm}$ on $[x^-, x^+]$.

Recall $k_1$ and $k_2$ as defined in (26) and consider the intersection points between the ellipse $K = 0$ and the lines $k_1 = 0$ and $k_2 = 0$. We define the following four quantities

$$x_0 := -2\mu_1 < 0 \quad \text{and} \quad y_0 := -2\mu_2 < 0, \tag{31}$$

$$x_1 := -\frac{2(r_2\mu_2 + \mu_1)}{1 + r_2^2 + 2\rho r_2} < 0 \quad \text{and} \quad y_2 := -\frac{2(r_1\mu_1 + \mu_2)}{1 + r_1^2 + 2\rho r_1} < 0. \tag{32}$$

These points are represented on Fig. 4 and satisfy the following properties:

- $K(x_0, 0) = k_2(x_0, 0) = 0$, $K(0, y_0) = k_1(0, y_0) = 0$.
- $\exists y_1 \in \mathbb{R}$ such that $K(x_1, y_1) = k_2(x_1, y_1) = 0$.
- $\exists x_2 \in \mathbb{R}$ such that $K(x_2, y_2) = k_1(x_2, y_2) = 0$.

We now define the curve $\mathcal{H}$, which is the boundary of the BVP established in Section 6.1.

$$\mathcal{H} = X^{\pm}([y^+, \infty)) = \{x \in \mathbb{C} \colon K(x, y) = 0 \text{ and } y \in [y^+, \infty)\}. \tag{33}$$

**Lemma 13** (*Hyperbola $\mathcal{H}$*)**.** *The curve $\mathcal{H}$ is a branch of the hyperbola with the following equation*

$$(\rho^2 - 1)x^2 + \rho^2 y^2 - 2(\mu_1 - \rho\mu_2)x = \mu_1(\mu_1 - 2\rho\mu_2). \tag{34}$$

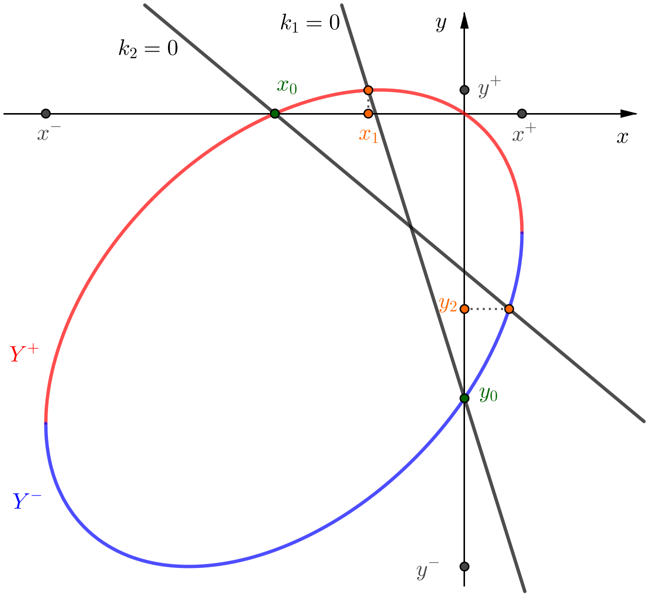



**Fig. 4.** The ellipse $K = 0$, the function $Y^-$ in blue, the function $Y^+$ in red, the two lines $k_1 = 0$ and $k_2 = 0$, the branch points $x^{\pm}$ and $y^{\pm}$, the points $x_0$ and $y_0$ in green, and the points $x_1$ and $y_2$ in orange. This figure is drawn for the following parameters: $\mu_1 = 2$, $\mu_2 = 3$, $\rho = -0.4$, $r_1 = 2$, $r_2 = 4$. (For a colored version of this figure, please see the electronic version of this article.).

*The curve $\mathcal{H}$ is symmetric with respect to the horizontal axis and is the right branch of the hyperbola when $\rho < 0$. Further, it is the left branch when $\rho > 0$ and it is a straight line when $\rho = 0$.*

**Proof.** A similar kernel has already been studied; we refer the reader to [21, Lemma 4] and [3, Lemma 9], where the equation of such a hyperbola is derived. □

Let $\mathcal{H}^+$ denote the part of the hyperbola $\mathcal{H}$ with positive imaginary part. We also define the domain $\mathcal{G}$ bounded by $\mathcal{H}$ and containing $x^+$. This is depicted in Fig. 5.

### 4.1. Meromorphic continuation

This section focuses on establishing the boundary value problem. We begin by meromorphically continuing the Laplace transform $\psi_1(x)$ (which converges for $x > 0$).

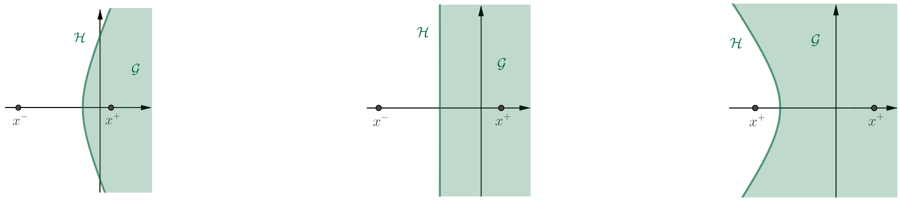
**Lemma 14** (*Meromorphic Continuation*). *By the formula*

$$\psi_1(x) = \frac{-k_2(x, Y^+(x))\psi_2(Y^+(x))}{k_1(x, Y^+(x))}, \tag{35}$$

*the Laplace transform $\psi_1(x)$ can be meromorphically continued to the set*

$$S := \{x \in \mathbb{C} : \Re x > 0 \text{ or } \Re Y^+(x) > 0\} \cup \{0\}, \tag{36}$$

*where the domain $\mathcal{G}$ and its boundary $\mathcal{H}$ are included in the set defined in (36). Then $\psi_1$ is meromorphic on $\mathcal{G}$ and is continuous on $\overline{\mathcal{G}}$.*




(a) $\rho < 0$ (b) $\rho = 0$ (c) $\rho > 0$

**Fig. 5.** Hyperbola $\mathcal{H}$ and domain $\mathcal{G}$.

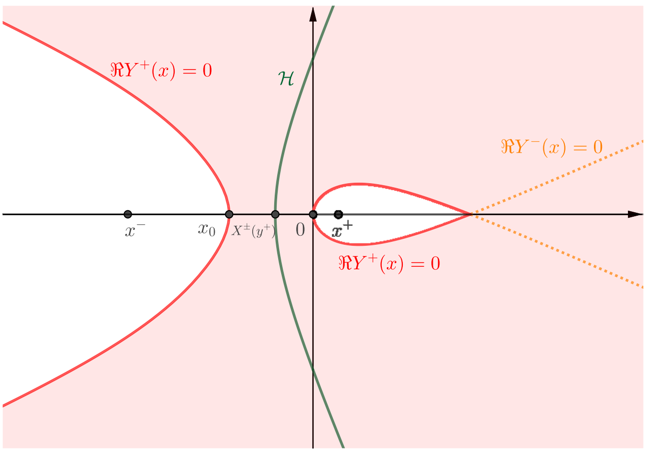



**Fig. 6.** The complex plane of $x$. The red curve of equation $\Re Y^+(x) = 0$ bounds the red domain $S_1 := \{x \in \mathbb{C} : \Re Y^+(x) > 0\}$. The orange dotted curve corresponds to the equation $\Re Y^-(x) = 0$. The domain $\mathcal{G}$ is bounded on the left by the green hyperbola $\mathcal{H}$, contains $x^+$ (see Fig. 5), and is included in $S = S_1 \cup S_2$, where $S_2 := \{x \in \mathbb{C} : \Re x > 0\}$. This figure is drawn for the parameters $\mu_1 = 2$, $\mu_2 = 3$, $\rho = -0.4$. (For a colored version of this figure, please see the electronic version of this article.).

**Proof.** The Laplace transforms $\psi_1(x)$ and $\psi_2(y)$ are analytic on $\{x \in \mathbb{C} \colon \Re x > 0\}$ and $\{y \in \mathbb{C} \colon \Re y > 0\}$, respectively. The functional equation (27) implies that for $(x, y)$ in the set $\widetilde{S} := \{(x, y) \in \mathbb{C}^2 : \Re x > 0,\ \Re y > 0 \text{ and } \psi(x, y) = 0\}$, we have

$$0 = k_1(x, y)\psi_1(x) + k_2(x, y)\psi_2(y). \tag{37}$$

The open connected set

$$S_1 := \{x \in \mathbb{C} \colon \Re Y^+(x) > 0\},$$

intersects the open set $S_2 := \{x \in \mathbb{C} : \Re x > 0\}$. For $x \in S_1 \cap S_2$, $(x, Y^+(x)) \in \widetilde{S}$; equation (37) implies that the continuation formula in (35) is satisfied for all $x \in S_1 \cap S_2$. Fig. 6 represents these sets. With $\psi_1(x)$ defined as in (35), we invoke the principle of analytic continuation and meromorphically extend $\psi_1$ to $S = S_1 \cup S_2$. Note that the inclusion of $\mathcal{G}$ in the set $S$ defined in (36) is similar to that in [21, Lemma 5]. This inclusion is depicted below in Fig. 6. □

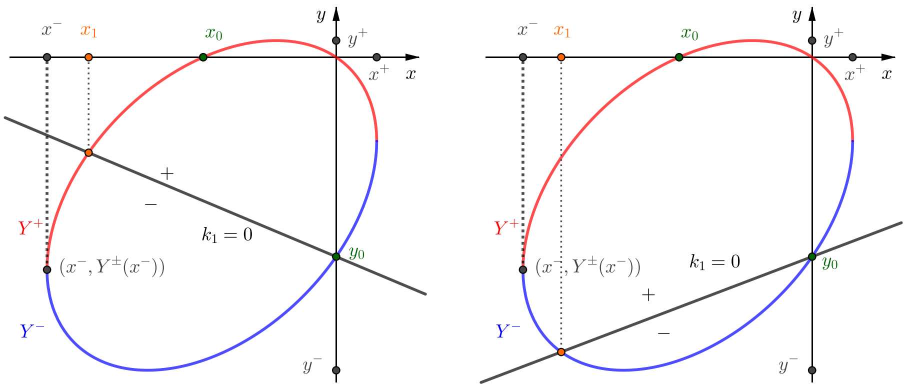



**Fig. 7.** On the left, we see that $k_1(x^-, Y^\pm(x_-)) < 0$ and that $x_1$ is a simple pole of $\psi_1$. On the right, we see that $k_1(x^-, Y^\pm(x^-)) > 0$ and that $\psi_1$ has no pole in $S$.

### 4.2. Poles and geometric conditions

**Lemma 15** (*Poles*). *On the set $S$ defined in (36), the Laplace transform $\psi_1$ has either one or two poles, as follows:*

- *(One pole:) If $k_1(x^-, Y^\pm(x^-)) \geqslant 0$, the point 0 is the unique pole of $\psi_1$ in $S$ and this pole is simple.*
- *(Two poles:) If $k_1(x^-, Y^\pm(x^-)) < 0$, the points 0 and $x_1$ (defined in (32)) are the only possible poles of $\psi_1$ in $S$ and these poles are simple; $x_1 \in S$ if and only if $x_1 > x_0$.*

*In addition, $\lim_{x\to 0} x\psi_1(x) = 1$. Further, the point $x_1$ is a pole of $\psi_1$ and belongs to the domain $\mathcal{G}$ if and only if $k_1(X^\pm(y^+), y^+) < 0$.*

**Proof.** The final value theorem for the Laplace transform, together with Theorem 4, imply that

$$\lim_{x\to 0} x\psi_1(x) = \lim_{u\to\infty} \mathbb{P}_{(u,0)}[T = \infty] = 1.$$

We may thus conclude that 0 is a simple pole. On the set $\{x \in \mathbb{C} : \Re x > 0\}$, $\psi_1$ is defined as a Laplace transform which converges (and thus has no poles). Therefore, with the exception of 0, the only possible poles in $S$ are the zeros of $k_1(x, Y^+(x))$, which are the zeros of the denominator of the continuation formula in (35). Straightforward calculations show that equation $k_1(x, Y^+(x)) = 0$ has either no roots or one (simple) root, and that this depends on the sign of $k_1(x^-, Y^\pm(x^-))$. When the root exists, it is $x_1$ (see (32)). The condition for the existence of this root is depicted in Fig. 7. It now only remains to remark that when $x_1$ is a pole, $x_1$ is in $\mathcal{G}$ if and only if $x_1 > X^\pm(y^+)$. The latter holds if and only if $k_1(X^\pm(y^+), y^+) < 0$ (see Fig. 8). □

Before turning to Lemma 16, we recall that the angles $\delta$, $\beta$ and $\theta$ were defined above in (3) and that $k_1$ was defined in (26).

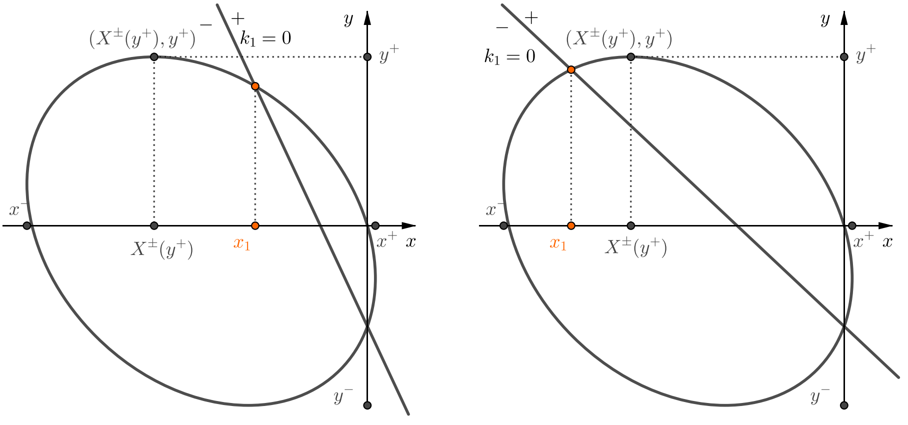



**Fig. 8.** On the left, we see that $k_1(X^{\pm}(y^+), y^+) < 0$ and that $x_1$ is in $\mathcal{G}$. On the right, we see that $k_1(X^{\pm}(y^+), y^+) > 0$ and that $x_1$ is not in $\mathcal{G}$.

**Lemma 16** (*Geometric Conditions*)**.** *The condition $k_1(x^-, Y^{\pm}(x^-)) > 0$ (resp. $= 0$ and $< 0$) is equivalent to*

$$2\delta - \theta < \pi,$$

*(resp. $= \pi$ and $> \pi$). The condition $k_1(X^{\pm}(y^+), y^+) > 0$ (resp. $= 0$ and $< 0$) is equivalent to*

$$2\delta - \theta + \beta < 2\pi,$$

*(resp. $= 2\pi$ and $> 2\pi$).*

**Proof.** By condition (1) and by the fact that the drift is positive, we have $0 < \theta < \beta < \delta < \pi$. By (3) and (29),

$$x^-/\mu_2 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\rho - \mu_1/\mu_2}{\sqrt{1-\rho^2}} - \sqrt{\left(\frac{\rho - \mu_1/\mu_2}{\sqrt{1-\rho^2}}\right)^2 + 1}\right) = \frac{-\cot(\theta) - \sqrt{\cot^2(\theta)+1}}{\sin(\beta)}. \tag{38}$$

We begin by proving the first equivalence for $\delta \geqslant \pi/2$. In this case we have

$$\begin{aligned}
k_1(x^-, Y^{\pm}(x^-)) > 0 &\Leftrightarrow \frac{1}{2}(r_2 x^- + Y^{\pm}(x^-)) + \rho x^- + \mu_2 > 0\\
&\Leftrightarrow r_2 + \rho < -\mu_2/x^- \text{ since } Y^{\pm}(x^-) = -\rho x^- - \mu_2 \text{ by (28) and (29)}\\
&\Leftrightarrow r_2 - \cos(\beta) < \sin(\beta)\left(\cot(\theta) + \sqrt{\cot^2(\theta)+1}\right)^{-1} \text{ by (38)}\\
&\Leftrightarrow -\cot(\delta)\left(\cot(\theta) + \sqrt{\cot^2(\theta)+1}\right) < 1\\
&\Leftrightarrow 0 < -\cot(\delta)\sqrt{\cot^2(\theta)+1} < 1 + \cot(\delta)\cot(\theta) \text{ since we assumed } \delta \geqslant \pi/2\\
&\Leftrightarrow \cot^2(\delta)(\cot^2(\theta)+1) < (1 + \cot(\delta)\cot(\theta))^2\\
&\Leftrightarrow 2\cot(\delta)\cot(\theta) - \cot^2(\delta) + 1 > 0\\
&\Leftrightarrow 2\sin(\delta)\cos(\delta)\cos(\theta) - (\cos^2(\delta) - \sin^2(\delta))\sin(\theta) > 0\\
&\Leftrightarrow \sin(2\delta)\cos(\theta) - \cos(2\delta)\sin(\theta) > 0\\
&\Leftrightarrow \sin(2\delta - \theta) > 0\\
&\Leftrightarrow 2\delta - \theta < \pi \text{ since } 0 < 2\delta - \theta < 2\pi.
\end{aligned}$$

It is straightforward to see that if $\delta < \pi/2$, then $2\delta - \theta < \pi$. Further, by (3), $\delta < \pi/2$ is equivalent to $r_2+\rho < 0$, which implies that $r_2+\rho < -\mu_2/x^-$. Therefore, $k_1(X^\pm(y^+), y^+) < 0$. This proves the first equivalence and thus the second equivalence is proved in exactly the same way, and thus the details are omitted. This concludes the proof. □

### *4.3. Absorption a symptotics along the axes*

In this section, we establish asymptotic results for the absorption probability (and escape probability) in the simpler case where the starting point is $(u, 0)$.

**Proposition 17** (*Absorption Asymptotics*)**.** *Let us assume that $x^- \in S$. For some constant $C$, the asymptotic behavior of $\mathbb{P}_{(u,0)}[T < \infty]$ as $u \to \infty$ is given by*

$$\mathbb{P}_{(u,0)}[T < \infty] \sim C \begin{cases} e^{ux_1} & \text{if } 2\delta - \theta > \pi, \\ u^{-\frac{3}{2}} e^{ux^-} & \text{if } 2\delta - \theta < \pi, \\ u^{-\frac{1}{2}} e^{ux^-} & \text{if } 2\delta - \theta = \pi. \end{cases}$$

**Proof.** The largest singularity of the Laplace transform of $\mathbb{P}_{(u,0)}[T < \infty]$ determines its asymptotics. We proceed by invoking a classical transfer theorem, see [11, Theorem 37.1]. This theorem says that if $a$ is the largest singularity of order $k$ of the Laplace transform (that is, the Laplace transform behaves as $(s-a)^{-k}$ up to additive and multiplicative constants in the neighborhood of $a$), then when $u \to \infty$, the probability $\mathbb{P}_{(u,0)}[T < \infty]$ is equivalent (up to a constant) to $u^{k-1}e^{au}$. The Laplace transform of $\mathbb{P}_{(u,0)}[T < \infty]$ is $1/x - \psi_1(x)$. By Lemma 15, the point 0 is not a singularity and the point $x_1$ is a simple pole. When that pole exists, the asymptotics are given by $Ce^{ux_1}$ for some constant $C$. When there is no pole, that is, when $k_1(x^-, Y^\pm(x^-)) \geqslant 0$, the largest singularity is given by the branch point $x^-$. The definition of $Y^+$ and (35) together imply that for some constants $C_i$ we have

$$\psi_1(x) \underset{x\to x^-}{=} \begin{cases} C_1 + C_2\sqrt{x - x^-} + \mathrm{O}(x - x^-) & \text{if } k_1(x^-, Y^\pm(x^-)) > 0, \\ \dfrac{C_3}{\sqrt{x - x^-}} + \mathrm{O}(1) & \text{if } k_1(x^-, Y^\pm(x^-)) = 0. \end{cases}$$

The proof is then completed by applying Lemma 16 and invoking the classical transfer theorem. □

**Remark 18** (*Asymptotics Along the Vertical Axis*)**.** In Proposition 17, we obtained the asymptotics for the absorption probability and for the escape probability along the horizontal axis. A similar study for $\psi_2$ would lead to the following asymptotics along the vertical axis. As $v \to \infty$,

$$\mathbb{P}_{(0,v)}[T < \infty] \sim C \begin{cases} e^{vy_2} & \text{if } 2\epsilon + \theta - \beta > \pi, \\ v^{-\frac{3}{2}} e^{vy^-} & \text{if } 2\epsilon + \theta - \beta < \pi, \\ v^{-\frac{1}{2}} e^{vy^-} & \text{if } 2\epsilon + \theta - \beta = \pi. \end{cases}$$

**Remark 19** (*Bivariate Asymptotics*)**.** The bivariate asymptotics of the absorption probability may be derived using the saddle point method and studying the singularities, see [13,19]. We would obtain some functions $a$, $b$, $c$, such that for $(u, v) = (r\cos(t), r\sin(t))$ in polar coordinates,

$$\mathbb{P}_{(u,v)}[T<\infty] \underset{r\to\infty}{\sim} a(t) r^{b(t)} e^{-c(t)r}.$$

Typically $b$ would take the value 0 or $-1/2$.

## 5. Product form and exponential absorption probability

In this section, we consider a remarkable geometric condition on the parameters characterizing the case where the absorption probability has a product form and is exponential. We call this new criterion the *dual skew symmetry* condition due to its natural connection with the famous skew symmetry condition studied by Harrison, Reiman and Williams [25,28], which characterizes the cases where the stationary distribution has a product form and is exponential. The *dual skew symmetry* condition gives a criterion for the solution to the partial differential equation of Proposition 11 (dual to that satisfied by the invariant measure) to be of product form. Theorem 20 below states a simple geometric criterion on the parameters for the absorption probability to be of product form; the absorption probability is then exponential.

**Theorem 20** (*Dual Skew Symmetry*)**.** *Let* $f(u,v) = \mathbb{P}_{(u,v)}[T<\infty]$ *be the absorption probability. The following statements are equivalent:*

1. *The absorption probability has a product form, i.e. there exist* $f_1$ *and* $f_2$ *such that*
$$f(u,v) = f_1(u) f_2(v);$$
2. *The absorption probability is exponential, i.e. there exist* $x$ *and* $y$ *in* $\mathbb{R}$ *such that*
$$f(u,v) = e^{ux+vy};$$
3. *The reflection vectors are in opposite directions, i.e.*
$$r_1 r_2 = 1;$$
4. *The reflection angles in the wedge satisfy* $\alpha = 1$*, i.e.*
$$\delta + \epsilon - \beta = \pi.$$

*In this case we have*

$$f(u,v) = e^{ux_1+vy_2}$$

*where* $x_1$ *and* $y_2$ *are given in* (32).

**Proof.** (1) $\Rightarrow$ (2): The Neumann boundary conditions in (23) imply that $f_1'(0)f_2(y) - r_1 f_1(0) f_2'(y) = 0$ and $-r_2 f_1'(u) f_2(0) + f_1(u) f_2'(0) = 0$. Solving these differential equations imply that $f_1$ and $f_2$ (and thus $f$) are exponential.
(2) $\Rightarrow$ (1): This implication is straightforward.
(2) $\Rightarrow$ (3): The Neumann boundary conditions in (23) imply that for all $v>0$, $xe^{vy} - r_1 y e^{vy} = 0$. Further, for all $u>0$, $-r_2 x e^{ux} + y e^{ux} = 0$. It follows that $r_1 = x/y$, $r_2 = y/x$, and thus $r_1 r_2 = 1$.
(3) $\Rightarrow$ (2): Let us define $f(u,v) = e^{ux_1+vy_2}$. We need to show that $f$ satisfies the partial differential equation of Proposition 11. This will imply that $f$ is the absorption probability. The fact that $r_1 = 1/r_2$, combined with (32), gives $r_1 = x_1/y_2$. This implies that $f$ satisfies the Neumann boundary conditions in (23). The limit values are satisfied because $f(0,0) = 1$

and $\lim_{(u,v)\to\infty} f(u,v) = 0$ for $x_1 < 0$ and $y_2 < 0$. It now only remains to show that $\mathcal{G}f = 0$. We now only need verify that $K(x_1, y_2) = 0$, see Fig. 9. By the definition of $y_2$ (see (32)), we have

$$K(x_1, y_2) = y_2\left(\frac{y_2}{2}\left(\left(\frac{x_1}{y_2}\right)^2 + 1 + 2\rho\frac{x_1}{y_2}\right) + \mu_1\frac{x_1}{y_2} + \mu_2\right)$$
$$= y_2\left(\frac{y_2}{2}\left(r_1^2 + 1 + 2\rho r_1\right) + \mu_1 r_1 + \mu_2\right) = 0.$$

(3) $\Leftrightarrow$ (4): The following equivalences hold:

$$r_1 r_2 = 1 \Leftrightarrow (\sin(\beta)/\tan(\delta) - \cos(\beta))\,(\sin(\beta)/\tan(\epsilon) - \cos(\beta)) = 1 \quad \text{by (3)}$$
$$\Leftrightarrow \frac{\sin(\beta)}{\tan(\epsilon)} = \frac{\tan(\delta)}{\sin(\beta) - \cos(\beta)\tan(\delta)} + \cos(\beta) = \frac{\tan(\delta)(1 - \cos^2(\beta)) + \cos(\beta)\sin(\beta)}{\sin(\beta) - \cos(\beta)\tan(\delta)}$$
$$\Leftrightarrow \tan(\epsilon) = \frac{\tan(\beta) - \tan(\delta)}{1 + \tan(\delta)\tan(\beta)}$$
$$\Leftrightarrow \tan(\epsilon) = \tan(\beta - \delta)$$
$$\Leftrightarrow \epsilon = \beta - \delta + \pi. \quad \square$$

**Remark 21** (*Standard and Dual Skew Symmetry*)**.** The standard skew symmetry condition for the matrix $\begin{pmatrix} 1 & -r_2 \\ -r_1 & 1 \end{pmatrix}$ is $2\rho = -r_1 - r_2$ or equivalently $\epsilon + \delta = \pi$. The standard skew symmetry condition for the dual matrix $\begin{pmatrix} r_2 & -1 \\ -1 & r_1 \end{pmatrix}$ defined in Section 1.5 is $2\rho = -1/r_1 - 1/r_2$ or equivalently $\epsilon + \delta - 2\beta = \pi$. Note that the dual skew symmetry condition obtained in Theorem 20 is different from these two conditions. Further properties of the *dual skew symmetry* condition will be explored in future work.

## 6. Integral expression of the Laplace transform $\psi_1$

In this section, we establish a boundary value problem (BVP) (for reference, see [8]) satisfied by the Laplace transform (Proposition 22). The section's key result is Theorem 30, which gives an explicit integral formula for the Laplace transform of the escape probability.

### 6.1. Carleman boundary value problem

We state a Carleman BVP satisfied by the Laplace transform $\psi_1$.

**Proposition 22** (*Carleman BVP*)**.** *The Laplace transform $\psi_1$ satisfies the following boundary value problem:*

*(i)* $\psi_1(x)$ *is meromorphic on* $\mathcal{G}$ *and is continuous on* $\overline{\mathcal{G}}$.
*(ii)* $\psi_1(x)$ *admits one or two poles in* $\mathcal{G}$. 0 *is always a simple pole and* $x_1$ *is a simple pole if and only if* $2\delta - \theta + \beta > 2\pi$.
*(iii)* $\lim_{x\to\infty} x\psi_1(x) = 0$.
*(iv)* $\psi_1$ *satisfies the boundary condition*

$$\psi_1(\overline{x}) = G(x)\psi_1(x), \quad \forall x \in \mathcal{H},$$

*where*

$$G(x) := \frac{k_1}{k_2}(x, Y^+(x))\frac{k_2}{k_1}(\overline{x}, Y^+(x)). \tag{39}$$

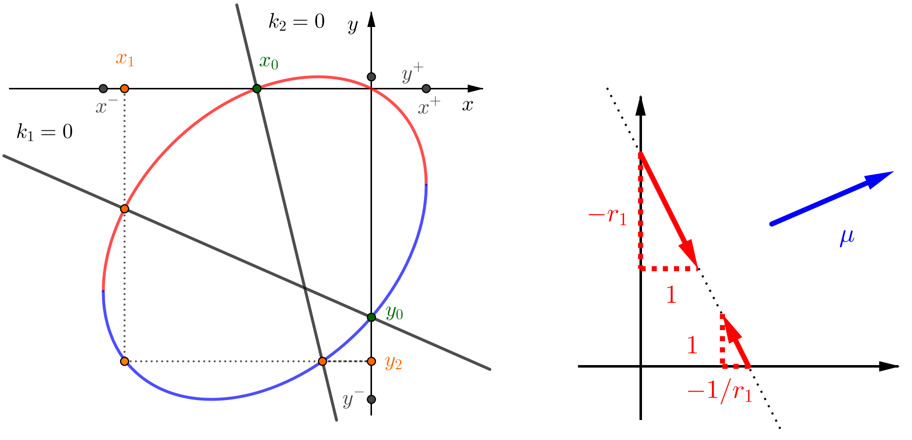



**Fig. 9.** Dual skew symmetry: on the left, we see that $K(x_2, y_2) = 0$; on the right, we see that condition $r_1 r_2 = 1$ implies that the reflection vectors are in opposite directions.

**Proof.** Statement (i) immediately follows from Lemma 14. Statement (ii) immediately follows from Lemmas 15 and 16. Statement (iii) follows from the initial value theorem for the Laplace transform, which implies that $\lim_{x\to\infty} x\psi_1(x) = \mathbb{P}_{(0,0)}[T = \infty] = 0$. To prove statement (iv), we recall the functional equation in (27). For $x \in \mathcal{H}$, we evaluate this equation for $(x, Y^+(x))$ and $(\overline{x}, Y^+(\overline{x}))$. By the definition of $Y^+$, we have $K(x, Y^+(x)) = K(\overline{x}, Y^+(\overline{x})) = 0$. By the definition of the hyperbola $\mathcal{H}$ in (34), we have that $Y^+(\overline{x}) = Y^+(x)$. This enables us to obtain the following system of equations

$$\begin{cases} 0 = k_1(x, Y^+(x))\psi_1(x) + k_2(x, Y^+(x))\psi_2(Y^+(x)), \\ 0 = k_1(\overline{x}, Y^+(x))\psi_1(\overline{x}) + k_2(\overline{x}, Y^+(x))\psi_2(Y^+(x)). \end{cases}$$

Solving this system of equations and eliminating $\psi_2(Y^+(x))$, we obtain the boundary condition in statement (iv). □

### 6.2. Gluing function

To solve the BVP, we need a conformal gluing function which glues together the upper and lower parts of the hyperbola. This conformal gluing function was introduced in [20,21]. For $a \geqslant 0$ and for $x \in \mathbb{C} \setminus (-\infty, -1]$, the generalized Chebyshev polynomial is defined as

$$T_a(x) := \cos(a \arccos(x)) = \frac{1}{2}\left((x + \sqrt{x^2 - 1})^a + (x - \sqrt{x^2 - 1})^a\right).$$

We define the angle

$$\beta := \arccos(-\rho).$$

We also define the functions

$$w(x) := T_{\frac{\pi}{\beta}}\left(\frac{2x - (x^+ + x^-)}{x^+ - x^-}\right), \tag{40}$$

and

$$W(x) := \frac{w(x) - w(X^{\pm}(y^+))}{w(x) - w(0)}.$$

We now recall a useful lemma from Franceschi and Raschel [21] for the conformal gluing function $W$.

**Lemma 23** (*Lemma 9, [21]*). *The function $W$ satisfies the following properties*

*(i) $W$ is holomorphic in $\mathcal{G} \setminus \{0\}$, continuous in $\overline{\mathcal{G}} \setminus \{0\}$ and bounded at infinity.*
*(ii) $W$ is bijective from $\mathcal{G} \setminus \{0\}$ to $\mathbb{C} \setminus [0, 1]$.*
*(iii) $W$ satisfies the gluing property on the hyperbola*

$$W(x) = W(\overline{x}), \quad \forall x \in \mathcal{H}.$$

*6.3. Index*

We proceed with some necessary notation. Let the angle $\Delta$ be the variation of the argument of $G(x)$ when $x$ lies on $\mathcal{H}^+$:

$$\Delta := [\arg G(x)]_{\mathcal{H}^+} = \left[\arg \frac{k_1}{k_2}(x, Y^+(x))\right]_{\mathcal{H}}.$$

Further, let $d$ be the argument of $G$ at the real point of the hyperbola $\mathcal{H}$:

$$d := \arg G(X^+(y^+)) \in (-\pi, \pi].$$

We define the index $\kappa$ as

$$\kappa := \left\lfloor \frac{d + \Delta}{2\pi} \right\rfloor.$$

The index shall prove useful to solving the boundary value problem in Proposition 22.

**Lemma 24.** *We have*

$$d = \begin{cases} 0 & \text{if } k_1(x^-, Y^{\pm}(x^-)) \neq 0 \text{ i.e. } 2\delta - \theta + \beta \neq 2\pi, \\ \pi & \text{if } k_1(x^-, Y^{\pm}(x^-)) = 0 \text{ i.e. } 2\delta - \theta + \beta = 2\pi, \end{cases}$$

*and*

$$\tan \frac{d + \Delta}{2} = \frac{(1 - (r_1 + 2\rho)(r_2 + 2\rho))\sqrt{1 - \rho^2}}{r_1 + r_2 + 3\rho - r_1 r_2 \rho - 2(r_1 + r_2)\rho^2 - 4\rho^3} = \tan(\epsilon + \delta + \beta).$$

*Note also that $\epsilon + \delta + \beta \geqslant 2\pi$ is equivalent to $1 - (r_1 + 2\rho)(r_2 + 2\rho) \leqslant 0$.*

**Proof.** The proof is (in each step) similar to the proof of Franceschi and Raschel [21, Lemma 13]. Firstly, note that the value of $d$ is obtained by the fact that $G(X^+(y^+)) = 1$ if $k_1(x^-, Y^{\pm}(x^-)) \neq 0$ and that $G(X^+(y^+)) = -1$ if $k_1(x^-, Y^{\pm}(x^-)) = 0$. Recall that, by definition, we have $\Delta = \lim_{\substack{x \to \infty \\ x \in \mathcal{H}^+}} \arg G(x) - d$ and that by (39) we have

$$G(x) = \frac{(\frac{1}{2}(r_2 + Y^+(x)/x) + \rho + \mu_2/x)(\frac{1}{2}(1 + r_1 Y^+(x)/\overline{x}) + \rho Y^+(x)/\overline{x} + \mu_1/\overline{x})}{(\frac{1}{2}(r_2 + Y^+(x)/\overline{x}) + \rho + \mu_2/\overline{x})(\frac{1}{2}(1 + r_1 Y^+(x)/x) + \rho Y^+(x)/x + \mu_1/x)}.$$

By (28), we compute the limit

$$\lim_{\substack{x\to\infty\\ x\in\mathcal{H}^+}} \frac{Y^+(x)}{x} = -\rho + i\sqrt{1-\rho^2},$$

from which we obtain

$$\begin{aligned} e^{i(\Delta+d)} &= \lim_{\substack{x\to\infty\\ x\in\mathcal{H}^+}} G(x) \\ &= \frac{(r_2+\rho+i\sqrt{1-\rho^2})(1-r_1\rho-2\rho^2-i(r_1+2\rho)\sqrt{1-\rho^2})}{(r_2+\rho-i\sqrt{1-\rho^2})(1-r_1\rho-2\rho^2+i(r_1+2\rho)\sqrt{1-\rho^2})} \\ &= \frac{(r_2+\rho)(1-r_1\rho-2\rho^2)+(r_1+2\rho)(1-\rho^2)+i(1-r_1r_2-2(r_1+r_2)\rho-4\rho^2)\sqrt{1-\rho^2}}{(r_2+\rho)(1-r_1\rho-2\rho^2)+(r_1+2\rho)(1-\rho^2)-i(1-r_1r_2-2(r_1+r_2)\rho-4\rho^2)\sqrt{1-\rho^2}}. \end{aligned}$$

We then see that

$$\tan\frac{d+\Delta}{2} = \frac{(1-r_1r_2-2(r_1+r_2)\rho-4\rho^2)\sqrt{1-\rho^2}}{(r_2+\rho)(1-r_1\rho-2\rho^2)+(r_1+2\rho)(1-\rho^2)} = \tan(\epsilon+\delta+\beta),$$

where the last equality follows from (3) and by straightforward calculation. The proof concludes by recalling the two following facts:

1. For $\alpha = \frac{\epsilon+\delta-\pi}{\beta} \geqslant 1$ and for $\epsilon, \delta$ and $\beta \in (0,\pi)$, we have that $-\pi < 2\beta-\pi \leqslant \epsilon+\delta+\beta-2\pi < \pi$.
2. By (3), $\sin(\epsilon+\delta+\beta)$ has the same sign as that $(r_1+2\rho)(r_2+2\rho)-1$, where $(r_1+2\rho)(r_2+2\rho)-1 = \sin(\epsilon+\delta+\beta)\frac{\sin(\beta)}{\sin(\epsilon)\sin(\delta)}$. □

We now proceed to state Lemma 25. For $1-(r_1+2\rho)(r_2+2\rho) \neq 0$, let us define

$$\widetilde{y} := 2\frac{\mu_2-\mu_1(r_2+2\rho)}{(r_1+2\rho)(r_2+2\rho)-1} = 2\mu_1\frac{\sin(\beta+\delta-\theta)\sin(\epsilon)}{\sin(\beta-\theta)\sin(\epsilon+\delta+\beta)}, \tag{41}$$

where the last equality holds by (3).

**Lemma 25.** *If $\widetilde{y}-y^+ \leqslant 0$ or if $1-(r_1+2\rho)(r_2+2\rho)=0$ then*

$$(G(x)=1 \text{ and } x\in\mathcal{H}) \Leftrightarrow x = X^{\pm}(y^+),$$

*and thus $d+\Delta \in (-2\pi, 2\pi)$. If $\widetilde{y}-y^+ > 0$ then*

$$(G(x)=1 \text{ and } x\in\mathcal{H}) \Leftrightarrow (x = X^{\pm}(y^+) \text{ or } x = X^{\pm}(\widetilde{y})),$$

*and thus $d+\Delta \in (-4\pi, 4\pi)$.*

**Proof.** Assume that $x\in\mathcal{H}$, where $x = a+ib$ for $a, b\in\mathbb{R}$ and $y = Y^{\pm}(x)$. Then by (39), $G(x)=1$ is equivalent to $\Im(k_1(a+ib,y)k_2(a-ib,y))=0$. Straightforward calculations yield

$$\Im(k_1(a+ib,y)k_2(a-ib,y)) = \frac{b}{4}\left[\frac{y}{2}((r_1+2\rho)(r_2+2\rho)-1)-2\mu_2+2\mu_1(r_2+2\rho)\right],$$

from which we may obtain that $G(x)=1$ is equivalent to $b=0$ or to $y=\widetilde{y}$. We conclude the proof by noting that

1. $b=0$ and $x\in\mathcal{H}$ together imply that $x = X^{\pm}(y^+)$, the latter being the only real point of the hyperbola.
2. By the definition of (33), $x\in\mathcal{H}$ and $y=\widetilde{y}$ imply that $\widetilde{y}\in[y^+,\infty)$. □

**Table 1**
Value of the index $\kappa$.

| | $\epsilon+\delta+\beta \geqslant 2\pi$ | $\epsilon+\delta+\beta < 2\pi$ |
|---|---|---|
| $2\delta-\theta+\beta > 2\pi$ | $\kappa=-1$ | $\kappa=-2$ |
| $2\delta-\theta+\beta \leqslant 2\pi$ | $\kappa=0$ | $\kappa=-1$ |

We continue with Lemma 26.

**Lemma 26.** *Assume that $2\delta-\theta+\beta > 2\pi$. Then $\widetilde{y} > y^+$ is equivalent to $\epsilon+\delta+\beta < 2\pi$.*

**Proof.** We first note that $2\delta-\theta+\beta > 2\pi$ implies that $\pi < \delta-\theta+\beta < 2\pi$, and thus $\sin(\delta-\theta+\beta) < 0$. Recall that we have previously seen that the conditions in (1) are equivalent to $\alpha \geqslant 1$ and $\delta > \beta$, $\epsilon > \beta$, and thus $\pi < \epsilon+\delta+\beta < 3\pi$. We employ the following steps to conclude the proof:

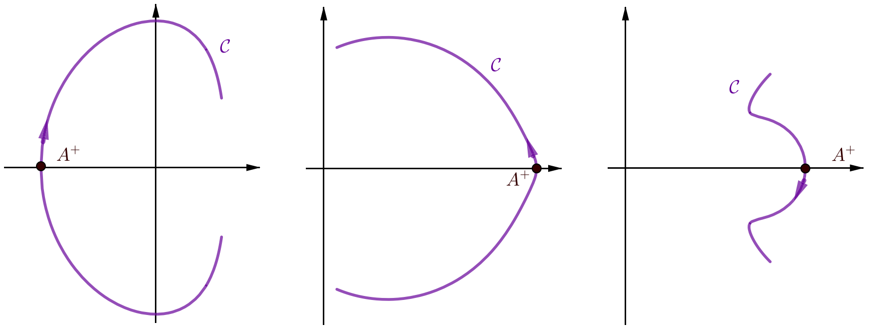
1. Assume that $\widetilde{y} > y^+$. Then for $y^+ > 0$, we have that $\widetilde{y} > 0$. Then by (41) we have that $\sin(\epsilon+\delta+\beta) < 0$ and thus $\epsilon+\delta+\beta < 2\pi$.
2. We now assume that $\epsilon+\delta+\beta < 2\pi$. Hence $\sin(\epsilon+\delta+\beta) < 0$. By hypothesis we have $\beta < \epsilon < 2\pi-\beta-\delta$. Employing (41), we may easily see that $\epsilon \mapsto \widetilde{y}$ is increasing for $\beta < \epsilon < 2\pi-\beta-\delta$. Replacing $\epsilon$ by $\beta$ in (41), we deduce that
$$\widetilde{y} > y_\delta := 2\mu_1 \frac{\sin(\beta+\delta-\theta)\sin(\beta)}{\sin(\beta-\theta)\sin(2\beta+\delta)}.$$
By hypothesis, we have that $\pi + \frac{\theta-\beta}{2} < \delta < 2\pi - 2\beta$. Note that $\delta \mapsto y_\delta$ is increasing in this interval. We then see that
$$\widetilde{y} > y_{\inf} := 2\mu_1 \frac{\sin(\beta+\pi+\frac{\theta-\beta}{2}-\theta)\sin(\beta)}{\sin(\beta-\theta)\sin(2\beta+\pi+\frac{\theta-\beta}{2})}.$$
Employing (30) and performing straightforward calculations, we obtain
$$\widetilde{y} - y^+ > y_{\inf} - y^+ = \mu_1 \frac{-2\sin(\frac{\beta-\theta}{2})\sin^2(\frac{\beta+\theta}{2})}{\sin(\beta-\theta)\sin(\epsilon+\delta+\beta)\sin(\beta)} > 0. \quad \square$$

Before stating the main lemma of this section, we introduce the following indicator variable $\chi$, which is associated with the results of Lemma 15 and Lemma 16.

$$\chi := \begin{cases} -1 & \text{if } 2\delta-\theta+\beta > 2\pi \Leftrightarrow x_1 \text{ is a pole of } \psi_1 \text{ in } \mathcal{G}, \\ 0 & \text{if } 2\delta-\theta+\beta \leqslant 2\pi \Leftrightarrow \psi_1 \text{ has no pole but 0 in } \mathcal{G}. \end{cases} \tag{42}$$

**Lemma 27** (*Index*). *The index $\kappa$ satisfies*
$$\kappa := \begin{cases} \chi & \text{if } \epsilon+\delta+\beta \geqslant 2\pi, \\ \chi - 1 & \text{if } \epsilon+\delta+\beta < 2\pi. \end{cases}$$
*The value of the index appears below in Table 1.*

**Remark 28** (*Index and Argument Principle*)**.** Notice that the index can take the values 0, −1 and −2 whereas in [21, Lemma 14] the index takes only the values 0 and −1. The difference




(a) If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta > 2\pi$, then $\kappa = -1$. (b) If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then $\kappa = 0$. (c) If $\epsilon+\delta+\beta < 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then $\kappa = -1$.

**Fig. 10.** When $\widetilde{y} - y^+ \leqslant 0$: a plot of the curve $\mathcal{C} := \{\frac{k_1}{k_2}(x, Y^+(x)) : x \in \mathcal{H}\}$ and the point $A^+ := \frac{k_1}{k_2}(X^+(y^+), y^+)$.

comes from the fact that $\psi_1$ can have two distinct poles while in [21] the Laplace transform has at most one simple pole. The index is inherently connected to number of zeros and poles of $\psi_1$. In the case of a closed curve, the argument principle implies that the index is equal to the number of zeros minus the number of poles counted with multiplicity of the function of the BVP. See Fomichov et al. [17, Lemma 6.9] which presents a case where the boundary of the BVP is a circle. In our case, the boundary is an (unbounded) hyperbola and $\psi_1$ is not meromorphic at infinity. Therefore we cannot directly apply the argument principle and the index $\kappa$ is not always equal to the opposite of the number of poles $\chi$.

**Proof.** The proof proceeds with two separate cases.

*Case I:* $\widetilde{y} - y^+ \leqslant 0$. In this case, by Lemma 25, we have that $d + \Delta \in (-2\pi, 2\pi)$ and that $G(x) \neq 1$ for all $x \in \mathcal{H}$ such that $x \neq X^\pm(y^+)$. Then $\kappa = 0$ or $-1$ depending on the sign of $d + \Delta$. This sign is given by the sign of $\arg G(x)$ when $x \in \mathcal{H}^+$ and $x \to X^\pm(y^+)$. Note that $x = a + ib \in \mathcal{H}^+$ and $y = Y^+(x)$. We then compute

$$k_1(a+ib, y)k_2(\overline{a+ib}, y) = k_1(a, y)k_2(a, y) + \frac{b^2}{4}(r_2 + 2\rho) - i\frac{b}{4}(1 - (r_1 + 2\rho)(r_2 + 2\rho))(y - \widetilde{y}).$$

Fig. 10 represents the curve $\mathcal{C} := \{\frac{k_1}{k_2}(x, Y^+(x)) : x \in \mathcal{H}\}$. It is useful to remark that $\arg \frac{k_1}{k_2}(x, Y^+(x)) = \arg k_1(x, Y^+(x))/k_2(\overline{x}, Y^+(x))$. We may thus deduce that

$$\begin{aligned}\operatorname{sgn}\arg G(x) &= \operatorname{sgn}\arg(k_1(a+ib, y)k_2(a-ib, y)) \\ &= \operatorname{sgn}\frac{-b(1 - (r_1 + 2\rho)(r_2 + 2\rho))(y - \widetilde{y})}{k_1(a, y)k_2(a, y) + \frac{b^2}{4}(r_2 + 2\rho)}.\end{aligned}$$

For $x \in \mathcal{H}^+$, we have $k_2(X^\pm(y^+), y^+) > 0$, $b > 0$. When $x \to X^\pm(y^+)$, we have that $b \to 0$ and $a \to X^\pm(y^+)$. Thus for $x \in \mathcal{H}^+$ and $x \to X^\pm(y^+)$,

$$\begin{aligned}\operatorname{sgn}\arg G(x) &= -\operatorname{sgn}(k_1(X^\pm(y^+), y^+)(1 - (r_1 + 2\rho)(r_2 + 2\rho))(y - \widetilde{y})) \\ &= -\operatorname{sgn}(2\delta - \theta + \beta - 2\pi)\operatorname{sgn}(\epsilon + \delta + \beta - 2\pi),\end{aligned}$$

where the last equality comes from Lemmas 16 and 24, as well as from the fact that in this case $y - \widetilde{y} > 0$ for $y > y^+$. This allows us to conclude the following

- If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta > 2\pi$, then for $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$, the sign of $\arg G(x)$ is negative. We may thus deduce that $\kappa = -1$, see Fig. 10(a).
- If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then for $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$, the sign of $\arg G(x)$ is positive. We may thus deduce that $\kappa = 0$, see Fig. 10(b).
- If $\epsilon+\delta+\beta < 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, then for $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$, the sign of $\arg G(x)$ is positive. We may thus deduce that $\kappa = -1$, see Fig. 10(c).

We pause to note that by Lemma 26 it is not possible to have $\epsilon+\delta+\beta < 2\pi$ and $2\delta-\theta+\beta > 2\pi$. This is because we have assumed $\widetilde{y} \leqslant y^+$.

*Case II:* $\widetilde{y} - y^+ > 0$. In this case, by Lemma 25 we have that $d+\Delta \in (-4\pi, 4\pi)$ and $(G(x) = 1 \text{ and } x \in \mathcal{H}) \Leftrightarrow (x = X^{\pm}(y^+) \text{ or } x = X^{\pm}(\widetilde{y}))$. Then $\kappa \in \{-2, -1, 0, 1\}$. To obtain the value of the index we study the curve $\mathcal{C} := \{\frac{k_1}{k_2}(x, Y^+(x)) : x \in \mathcal{H}\}$. By straightforward calculations we see that $\widetilde{A} := \frac{k_1}{k_2}(X^{\pm}(\widetilde{y}), \widetilde{y})$ is positive. The study of the sign of the real and the imaginary parts of $\frac{k_1}{k_2}(x, Y^+(x))$ for $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$ gives the value of $\kappa$. Following the same logic as that of Case I above, we see that the real part of $\frac{k_1}{k_2}(x, Y^+(x))$ for $x \in \mathcal{H}^+$ and $x \to X^{\pm}(y^+)$ has the same sign as $-(2\delta-\theta+\beta-2\pi)$. Further, the imaginary part has the same sign that $-(\epsilon+\delta+\beta-2\pi)$. We may then conclude as follows:

- If $\epsilon+\delta+\beta < 2\pi$ and $2\delta-\theta+\beta > 2\pi$, $\kappa = -2$, see Fig. 11(a).
- If $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, $\kappa = 0$, see Fig. 11(b).
- If $\epsilon+\delta+\beta < 2\pi$ and $2\delta-\theta+\beta \leqslant 2\pi$, $\kappa = -1$, see Fig. 11(c).

Note that by Lemma 26 it is not possible to have $\epsilon+\delta+\beta \geqslant 2\pi$ and $2\delta-\theta+\beta > 2\pi$. This is because we have assumed that $\widetilde{y} > y^+$. □

We now state a technical lemma which shall be invoked in Section 7.

**Lemma 29.** *The following equality holds*

$$\left(-\frac{d+\Delta}{2\pi} + \chi - 1\right)\frac{\pi}{\beta} = -\alpha - 1.$$

**Proof.** First, we recall by Lemma 24 that

$$\tan\frac{d+\Delta}{2} = \tan(\epsilon+\delta+\beta).$$

For $\alpha = \frac{\epsilon+\delta-\pi}{\beta} \geqslant 1$ and $\epsilon, \delta$ and $\beta \in (0, \pi)$,

$$2\beta - \pi \leqslant \epsilon+\delta+\beta-2\pi < \pi.$$

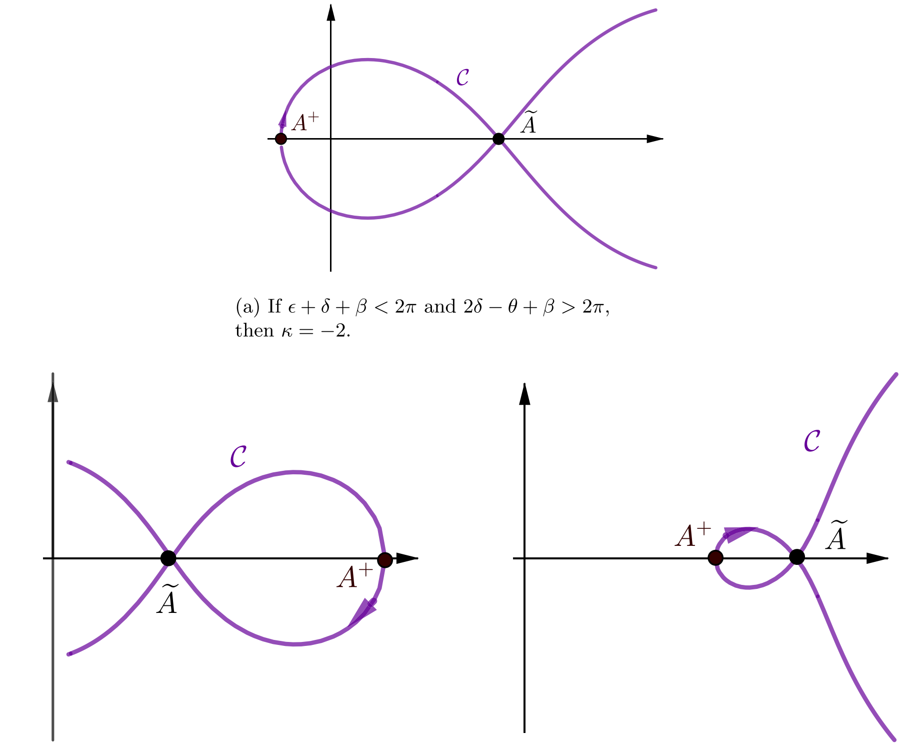
Further, recall that by definition, $\kappa = \lfloor \frac{d+\Delta}{2\pi} \rfloor$.

We now consider two cases for the value of $\epsilon+\delta+\beta-2\pi$. The first case considers $\epsilon+\delta+\beta-2\pi \geqslant 0$. In this case,

$$\frac{d+\Delta}{2} = \begin{cases} \epsilon+\delta+\beta-2\pi & \text{if } \frac{d+\Delta}{2\pi} \geqslant 0 \text{ i.e. } \kappa = 0, \\ \epsilon+\delta+\beta-3\pi & \text{if } \frac{d+\Delta}{2\pi} < 0 \text{ i.e. } \kappa = -1. \end{cases}$$

By Lemma 27, we have $\kappa = \chi$. We may thus deduce that

$$\frac{d+\Delta}{2} = \epsilon+\delta+\beta+(\chi-2)\pi.$$




(a) If $\epsilon + \delta + \beta < 2\pi$ and $2\delta - \theta + \beta > 2\pi$, then $\kappa = -2$.

(b) If $\epsilon + \delta + \beta \geqslant 2\pi$ and $2\delta - \theta + \beta \leqslant 2\pi$, then $\kappa = 0$.

(c) If $\epsilon + \delta + \beta < 2\pi$ and $2\delta - \theta + \beta \leqslant 2\pi$, then $\kappa = -1$.

**Fig. 11.** When $\widetilde{y} - y^+ > 0$: a plot of the curve $\mathcal{C} := \{\frac{k_1}{k_2}(x, Y^+(x)) : x \in \mathcal{H}\}$, the point $A^+ := \frac{k_1}{k_2}(X^+(y^+), y^+)$, and the point $\widetilde{A} := \frac{k_1}{k_2}(X^+(\widetilde{y}), \widetilde{y})$.

The second case considers $\epsilon + \delta + \beta - 2\pi < 0$. In this case,

$$\frac{d + \Delta}{2} = \begin{cases} \epsilon + \delta + \beta - 2\pi & \text{if } -\pi \leqslant \frac{d+\Delta}{2\pi} < 0 \text{ i.e. } \kappa = -1, \\ \epsilon + \delta + \beta - 3\pi & \text{if } -2\pi \leqslant \frac{d+\Delta}{2\pi} < -\pi \text{ i.e. } \kappa = -2. \end{cases}$$

By Lemma 27, we have $\kappa = \chi - 1$. We may thus deduce that

$$\frac{d + \Delta}{2} = \epsilon + \delta + \beta + (\chi - 2)\pi.$$

Thus, in both cases we have

$$\left(-\frac{d + \Delta}{2\pi} + \chi - 1\right)\frac{\pi}{\beta} = (-\epsilon - \delta - \beta - (\chi - 2)\pi + \chi\pi - \pi)\frac{1}{\beta} = -\alpha - 1.$$

This concludes the proof. $\square$

*6.4. Solution of the BVP*

The following theorem gives an explicit integral formula for the Laplace transform of the escape probability $\psi_1$.

**Theorem 30** (*Explicit Expression for* $\psi_1$)**.** *The Laplace transform* $\psi_1$ *is given for* $x \in \mathcal{G}$ *by*

$$\psi_1(x) = \frac{w'(0)}{w(x)-w(0)}\left(\frac{w(0)-w(x_1)}{w(x)-w(x_1)}\right)^{-\chi} \times \exp\left(\frac{1}{2i\pi}\int_{\mathcal{H}^+}\log G(t)\left[\frac{w'(t)}{w(t)-w(x)}-\frac{w'(t)}{w(t)-w(0)}\right]\mathrm{d}t\right), \tag{43}$$

*where* $x_1$ *is defined in* (32), $G$ *is defined* (39), $w$ *is defined in* (40), $\chi$ *is defined in* (42) *and* $\mathcal{H}$ *is defined in* (34).

**Remark 31.** We now provide some remarks about Theorem 30.

- The poles 0 and $x_1$ in Lemma 15 can be easily visualized by Theorem 30. The indicator variable $\chi$ defined in (42) indicates clearly whether or not the pole $x_1$ is in $\mathcal{G}$.
- A symmetrical result holds for $\psi_2$. Using the functional equation (27) we obtain an explicit formula for $\psi$. By inverting this Laplace transform we obtain the escape probability, which constitutes the main motivation for our work.
- It is still possible to deduce some concrete results from the integral formula obtained in Theorem 30. In Section 7 we derive a very explicit and simple expression for the asymptotics of the escape probability at the origin.
- Similar to [6, Thm 2.3, §9.1], we can show that $\psi_1$ is differentially algebraic if $\beta \in \pi\mathbb{Q}$. When $\psi_1$ is differentially algebraic, it satisfies a differential equation, allowing us to deduce a polynomial recurrence relation for the moments of the escape/absorption probability. See [6, §6.3] which gives an explicit example for the SRBM stationary distribution in the recurrent case.
- The methods and techniques employed to prove this theorem are inspired by those used to study random walks in the quarter plane ([15]).

**Proof.** Let

$$\widetilde{\psi}_1(y) := \frac{(y-W(x_1))^{-\chi}}{(y-1)^{1+\kappa-\chi}}\psi_1 \circ W^{-1}(y).$$

Proposition 22, Lemma 23 and Lemma 27 together imply that

- $\widetilde{\psi}_1$ is analytic on $\mathbb{C}\setminus[0,1]$.
- $\widetilde{\psi}_1(y) \sim_\infty c y^{-\kappa}$ for some constant $c$.
- $\widetilde{\psi}_1(1) = 0$.
- For $y \in [0,1]$, $\widetilde{\psi}_1$ satisfies the boundary condition

  $$\widetilde{\psi}_1^+(y) = \widetilde{G}(y)\widetilde{\psi}_1^-(y),$$

  where $\widetilde{\psi}_1^+(y)$ is the left limit and $\widetilde{\psi}_1^-(y)$ is the right limit of $\widetilde{\psi}_1$ on $[0,1]$, $(W^{-1})^-$ is the right limit of $W^{-1}$ on $[0,1]$, and $\widetilde{G}(y) = G\circ(W^{-1})^-(y)$.

We now define

$$\widetilde{S}(y) := (y-1)^{-\kappa}\exp\left(\frac{1}{2i\pi}\int_0^1\frac{\log\widetilde{G}(u)}{u-y}\right).$$

Following the classical boundary theory results in [15, (5.2.24) and Theorem 5.2.3], the above function is analytic and does not cancel on $\mathbb{C}\setminus[0,1]$ and is such that $\widetilde{S}(y)\sim_\infty c'y^{-\kappa}$ for some constant $c'$. Furthermore, for $y\in[0,1]$, it satisfies the boundary condition

$$\widetilde{S}^+(y)=\widetilde{G}(y)\widetilde{S}^-(y),$$

where $\widetilde{S}^+(y)$ is the left limit and $\widetilde{S}^-(y)$ is the right limit of $\widetilde{S}$ on $[0,1]$. By the properties of $\widetilde{\psi}_1$ and $\widetilde{S}$ detailed above, the function $\widetilde{\psi}_1/\widetilde{S}$ is analytic on $\mathbb{C}$ and bounded at infinity. Therefore there must exist a constant $C$ such that

$$\widetilde{\psi}_1(y)=C\widetilde{S}(y).$$

Invoking the definition of $\widetilde{\psi}_1$, we have that

$$\frac{(W(x)-W(x_1))^{-\chi}}{(W(x)-1)^{1+\kappa-\chi}}\psi_1(x)=C(W(x)-1)^{-\kappa}\exp\left(\frac{1}{2i\pi}\int_0^1\frac{\log\widetilde{G}(u)}{u-W(x)}\right).\tag{44}$$

Noting that

$$W(x)-1=\frac{w(0)-w(X^\pm(y^+))}{w(x)-w(0)}\quad\text{and}\quad W(x)-W(x_1)=\frac{w(x)-w(x_1)}{w(x)-w(0)}\frac{w(X^\pm(y^+))-w(0)}{w(x_1)-w(0)},$$

and making a change of variable $u=w(t)$ in the integral in (44), we obtain for some constant $C'$

$$\psi_1(x)=C'\left(\frac{1}{w(x)-w(0)}\right)\left(\frac{1}{w(x)-w(x_1)}\right)^{-\chi}\exp\left(\frac{1}{2i\pi}\int_{\mathcal{H}^+}\log G(t)\frac{w'(t)}{w(t)-w(x)}\mathrm{d}t\right).$$

The final value theorem for the Laplace transform gives

$$\lim_{x\to0}x\psi_1(x)=\lim_{u\to\infty}\mathbb{P}_{(u,0)}[T=\infty]=1.$$

This enables us to compute the constant

$$C'=w'(0)\,(w(0)-w(x_1))^{-\chi}\exp\left(\frac{-1}{2i\pi}\int_{\mathcal{H}^+}\log G(t)\frac{w'(t)}{w(t)-w(0)}\mathrm{d}t\right),$$

which gives us (43), completing the proof. □

## 7. Asymptotics of the escape probability at the origin

In this section we use the explicit expression in Theorem 30 to obtain the asymptotics of the escape probability at the origin. We begin by computing the asymptotics of $\psi_1$ at infinity.

**Lemma 32** (*Asymptotics of $\psi_1$*)**.** *Let $\alpha$ be defined as in (4). For ease of notation, allow $C$ to be a constant which may change from one line to the next. For some positive constant $C$,*

$$\psi_1(x)\underset{x\to\infty}{\sim}Cx^{-\alpha-1}.$$

*A symmetrical result holds for $\psi_2$. That is, for some positive constant $C$,*

$$\psi_2(y)\underset{y\to\infty}{\sim}Cy^{-\alpha-1}.$$

**Proof.** This proof follows the same steps as those of Franceschi and Raschel [21, Prop 19]. The key idea is to invoke [15, (5.2.20)], which states that

$$\exp\left(\frac{1}{2i\pi}\int_0^1\frac{\log\widetilde{G}(u)}{u-y}\right)\underset{y\to1}{\sim}C(y-1)^{\frac{d+\Delta}{2\pi}}.$$

Recall that $w(x) \underset{x\to\infty}{\sim} Cx^{\frac{\pi}{\beta}}$ and that $W(x) - 1 \underset{x\to\infty}{\sim} Cx^{-\frac{\pi}{\beta}}$. The explicit expressions of $\psi_1$ obtained in (43) and in (44) imply that

$$\psi_1(x) \underset{x\to\infty}{\sim} Cx^{(-\frac{d+\Delta}{2\pi}+\chi-1)\frac{\pi}{\beta}}.$$

The proof concludes by invoking Lemma 29, which states that $\left(-\frac{d+\Delta}{2\pi} + \chi - 1\right)\frac{\pi}{\beta} = -\alpha - 1$. □

**Lemma 33** (*Asymptotics of* $\psi$). *Let* $\alpha$ *defined as in* (4). *For* $t \in [0, \frac{\pi}{2}]$ *and some positive constant* $C_t$,

$$\psi(r\cos t, r\sin t) \underset{r\to\infty}{\sim} C_t r^{-\alpha-2}.$$

**Proof.** The result is immediate from the functional equation (27) and from Lemma 32. □

**Proposition 34** (*Asymptotics at the Origin*). *For positive constants* $c_0$ *and* $c_1$ *we have the following asymptotics*

$$\mathbb{P}_{(u,0)}[T = \infty] \underset{u\to 0}{\sim} c_0 u^\alpha \quad \text{and} \quad \mathbb{P}_{(0,v)}[T = \infty] \underset{v\to 0}{\sim} c_1 v^\alpha.$$

**Proof.** The result follows by combining the asymptotic results of $\psi_1$ and $\psi_2$ at infinity that we computed in Lemma 32 with the reciprocal of the result in [11, Thm 33.3].[2] We begin by denoting $g(u) := \mathbb{P}_{(u,0)}[T = \infty]$. Then, by definition, $\psi_1(x) = \int_0^\infty e^{-xu} g(u)\mathrm{d}u$. As $\psi_1(x)$ has no singularities greater than 0, for every $A > 0$, the inverse Laplace transform gives

$$g(u) = \frac{1}{2i\pi}\int_{A-i\infty}^{A+i\infty} e^{ux}\psi_1(x)\mathrm{d}x.$$

By Lemma 32, we have $\psi_1(x) = \frac{C+\eta(x)}{x^{\alpha+1}}$, where $\eta$ is a function such that $\lim_\infty \eta = 0$. Recalling that the Laplace transform of $u^\alpha/\Gamma(\alpha+1)$ is $x^{-\alpha-1}$ and performing a change of variables $s = ux$, we obtain

$$\begin{aligned} g(u) &= \frac{1}{2i\pi}\int_{A-i\infty}^{A+i\infty} e^{ux}\frac{C+\eta(x)}{x^{\alpha+1}}\mathrm{d}x \\ &= u^\alpha\left(\frac{C}{\Gamma(\alpha+1)} + \frac{1}{2i\pi}\int_{Au-i\infty}^{Au+i\infty} e^s\frac{\eta(s/u)}{s^{\alpha+1}}\mathrm{d}s\right). \end{aligned}$$

It remains to show that the last integral tends to 0 when $u \to 0$. To do so, consider $\epsilon > 0$ arbitrarily small. Then there exists $B > 0$ sufficiently large such that $\eta(x) < \epsilon$ for all $|x| > B$. For all $u$ such that $u < 1/B$, let us consider $A := 1/u$. This gives

$$\left|\frac{1}{2i\pi}\int_{Au-i\infty}^{Au+i\infty} e^s\frac{\eta(s/u)}{s^{\alpha+1}}\mathrm{d}s\right| < \frac{\epsilon}{2i\pi}\int_{1-i\infty}^{1+i\infty}\frac{1}{s^{\alpha+1}}\mathrm{d}s,$$

where the last integral converges for $\alpha \geqslant 1$. The proof concludes by letting $\epsilon$ tend towards 0. □

**Theorem 35** (*Asymptotics at the Origin*). *For* $t \in (0, \frac{\pi}{2})$ *and some positive constant* $c_t$ *we have*

$$\mathbb{P}_{(r\cos t, r\sin t)}[T = \infty] \underset{r\to 0}{\sim} c_t r^\alpha.$$

[2] Doetsch [11, Thm 33.3] establishes that if for some constant $a$ a function is equivalent to $u^a$ at 0, then at infinity, its Laplace transform is equivalent (up to a multiplicative constant) to $x^{-a-1}$.

**Proof.** This proof follows directly from the asymptotics of the double Laplace transform $\psi$ computed in Lemma 33. Recall the result used in the proof of Proposition 34 linking the asymptotics of a function at 0 to the asymptotics of its Laplace transform at infinity. The only necessary modification is to apply this result with a polar coordinate transformation. The desired asymptotics then follow with nearly identical calculations. □

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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