Reflected Brownian motion in a cone: the transient case

Persistence, escape and absorption probabilities,

Green's functions and Martin's boundary

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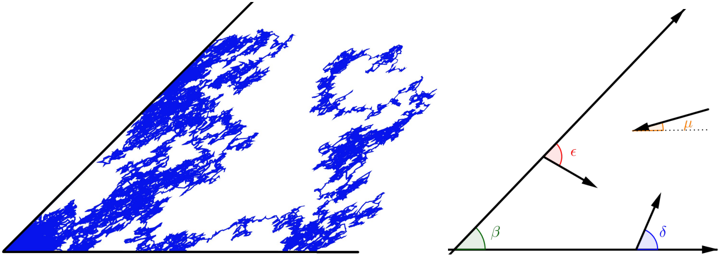
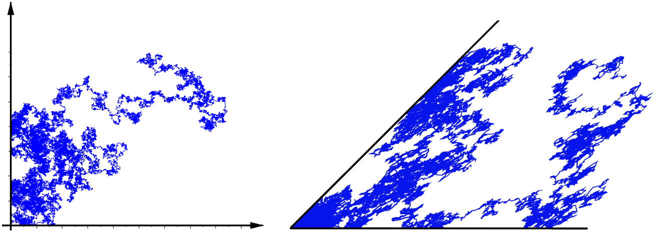
Introduction

Obliquely Reflected Brownian Motion in a Cone



- Oblique reflections constant along the edges
- Drift

From quadrant to cones



linear transform

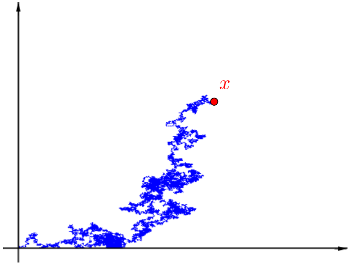
quadrant $\longleftrightarrow$ **cone**

continuous case (random walks) $\neq$ discrete case (Brownian)

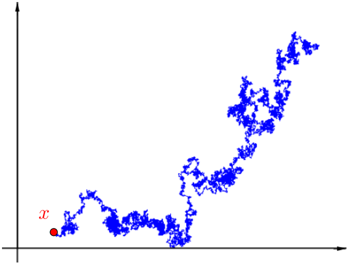
Persistence and absorption probability

♯ Goal 1

- Study the **absorption probability** at the apex of the cone



Absorption

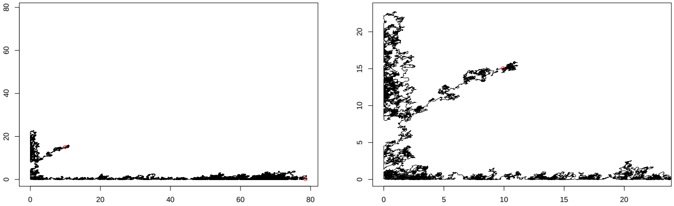


Persistence

Escape probability

♯ Goal 2

- Study the **escape probability along an axis**



Recurrence/Transience

According to the parameters of the model Z_t is :

- **Recurrent** (stationary distribution /invariant measure)

$\hookrightarrow$ Proportion of time that the process spends in the set A

$$\pi(A) = \lim_{t \to \infty} \mathbb{E}\left[\frac{1}{t}\int_0^t \mathbb{1}_A(Z_u)\mathrm{d}u\right]$$

- **Transient** $Z_t \to \infty$ (Green measures)

$\hookrightarrow$ Quantity of time that the process spends in A starting from z_0

$$g^{z_0}(A) = \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_u)\mathrm{d}u\right] = \int_0^\infty p_t(z_0, A)\mathrm{d}t$$

$\sharp$ Goal 3

- Study **Green's functions**
- Study **Martin boundary** and harmonic functions

Analytic approach

- **Probabilistic questions**

- **Analytic methods and tools**
  - Kernel functional equation (generating function, Laplace transform)
  - Boundary value problem (Carleman, Riemann-Hilbert, Sokhotski–Plemelj)
  - Analytic combinatorics (singularity, transfer lemmas, saddle point method, Tutte's invariants)

- **Results**
  - Exact expressions (countour integrals, hypergeometric functions)
  - Asymptotics and Martin boundary (harmonic functions)
  - Algebraic nature (rational, algebraic, DF, DA)

Approach developed in the **discrete** setting (random walks in the quadrant) by G. Fayolle et V. Malyshev in the seventies

$\hookrightarrow$ **continuous** setting (Brownian)

Plan

1. Obliquely reflected Brownian motion in the quadrant
   - Absorption at the apex
   - Escape along an axis
   - Green's functions
2. Analytic approach
   - Functional equation
   - Boundary value problem
   - Explicit expression
3. Martin and Poisson boundary
   - Asymptotics
   - Saddle point method and transfer lemma
   - Martin and Poisson boundary, harmonic functions

Plan

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Local time

- B_t Brownian motion
- Define the **local time** by

$$L_t^a = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{[a, a+\epsilon]}(B_s) \mathrm{d}s$$

$\hookrightarrow$ density of time spends in a before t

Occupation time formula

$$\int_0^t f(B_s) \mathrm{d}s = \int_{\mathbb{R}} f(a) L_t^a \mathrm{d}a$$

- $|B_t|$: **Reflected Brownian motion** in dimension 1 = absolute value of B_t
- **Itô's formula**

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s$$

$\hookrightarrow$ Applied to $f = |\cdot|$, $f' = \mathrm{sgn}$ et $f'' = 2\delta_0$

$$\Rightarrow |B_t| = 0 + \underbrace{\int_0^t \mathrm{sgn}(B_s)\mathrm{d}B_s}_{\substack{\text{Brownian motion} \\ W_t}} + \underbrace{\frac{1}{2}\int_0^t 2\delta_0(B_s)\mathrm{d}s}_{\substack{\int_{\mathbb{R}} \delta_0(a) L_t^a \mathrm{d}a = L_t^0 \\ \text{occupation time formula}}}$$

Tanaka's formula

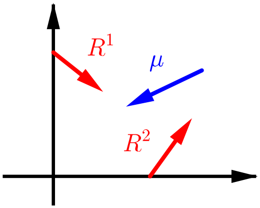
$$|B_t| = W_t + L_t^0$$

- W_t Brownian motion
- L_t^0 local time in 0

Obliquely reflected Brownian motion with drift in $\mathbb{R}_+^2$

Model parameters

- W_t Planar Brownian motion, covariance $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$
- $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ drift
- $R = (R_1, R_2) = \begin{pmatrix} 1 & r_2 \\ r_1 & 1 \end{pmatrix}$ reflection matrix



Definition (Semimartingale Reflected Brownian Motion)

We define a SRBM starting from z_0 by

$$Z_t = z_0 + W_t + \mu t + RL_t \in \mathbb{R}_+^2$$

where L_t^i is a continuous and increasing process which increase only when the process hit an axis.

$\hookrightarrow$ L_t is the **local time** on the axis $\hookrightarrow$ **Skorokhod** problem

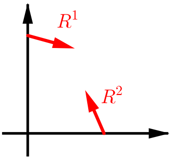
Obliquely reflected Brownian motion with drift in $\mathbb{R}_+^2$

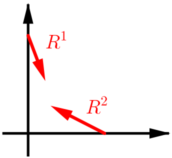
Theorem (Reiman, Taylor, Williams, 1988 et 1993)

Such a process Z_t **exists (or persist)** for all $t \geqslant 0$
$\Leftrightarrow r_1, r_2 > 0$ or $1 - r_1 r_2 > 0$
$\Leftrightarrow \exists$ **convex combination** of R^1 and R^2 which belongs to $\mathbb{R}_+^2$.

Absorption at the apex

Otherwise, Z_t can reach the apex and get stuck : **absorption**





- Existence $\forall t$

Persistence

- Do not exist $\forall t$

Absorption at the apex

Persistence and absorption

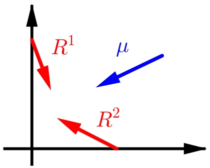
- **Assume that** $r_1, r_2 \leqslant 0$ and $1 - r_1 r_2 \leqslant 0$
  i.e. **absorption** is possible (persistence $\forall t$ is not certain)
- $p_A(z_0)$ absorption probability
  $\hookrightarrow$ function of the starting point z_0

Absorption and persistence probability (2021, Ernst, F.)

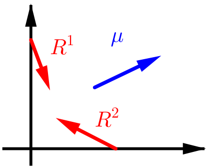
Either the process is absorbed in a finite time or it escapes to infinity.

$$p_A(z_0) = 1 - \mathbb{P}_{(u,v)}\left(\lim_{t\to\infty} Z_t = \infty\right).$$

- If $\mu < 0$ then $p_A(z_0) = 1$



- If $\mu > 0$ then $p_A(z_0) \in (0, 1)$

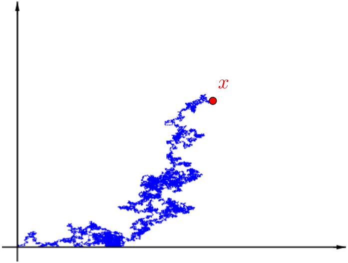


Goal $\sharp$1

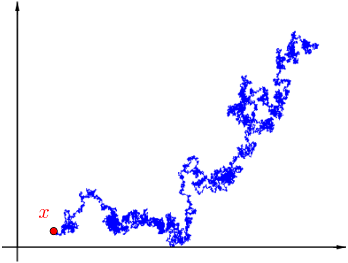
Compute $p_A(z_0)$

Escape and absorption

x is the starting point



Absorption



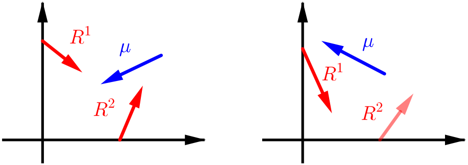
Escape & Persistence

Proposition (D. Hobson et L. Rogers, 1993)

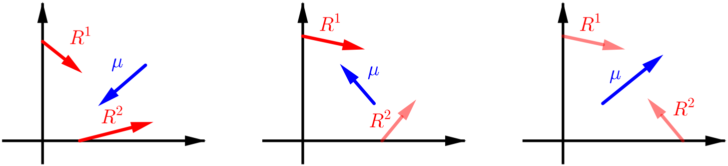
- *Recurrent* $\Leftrightarrow 1 - r_1 r_2 > 0, \quad \mu_1 - r_2\mu_2 \leqslant 0, \quad \mu_2 - r_1\mu_1 \leqslant 0$
- *Transient otherwise*

▶ **Competition between drift and reflection vectors**

Recurrent case



Transient case



Escape along an axis

Transient case, two ways of escape to infinity :

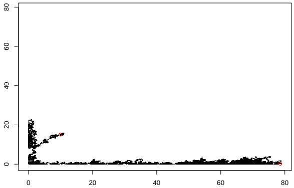
- The two coordinates tend to infinity $\Leftrightarrow \mu_1 > 0$ et $\mu_2 > 0$
- Along one of the axes $\Leftrightarrow \mu_1 < 0$ ou $\mu_2 < 0$
  - $p_1(z_0)$ escape probability along the horizontal axis
  - $p_2(z_0)$ escape probability along the vertical axis

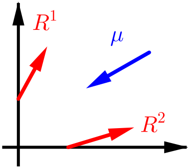
$\hookrightarrow$ Most of the time $p_1(z_0) = 0$ or 1

Escape probability (2020+, Fomichov, F., Ivanovs)

Assume that $\mu < 0$, $r_1, r_2 > 0$, $r_1 r_2 - 1 > 0$, $\mu_1 - r_2\mu_2 > 0$, $\mu_2 - r_1\mu_1 > 0$

$$p_1(z_0) + p_2(z_0) = 1 \quad \text{et} \quad p_1(z_0) \text{ et } p_2(z_0) \in (0,1)$$





Goal ♯2

Compute $p_1(z_0)$ and $p_2(z_0)$

Transient case

Green's functions

$$g^{z_0}(z)\mathrm{d}z = \mathbb{E}_{z_0} \int_0^\infty \mathbb{1}_{\mathrm{d}z}(Z_u)\mathrm{d}u = \int_0^\infty p_t(z_0, \mathrm{d}z)\mathrm{d}t$$

Green's functions ont the boundary $h_i^{z_0}(z)$

$$h_i^{z_0}(z)\mathrm{d}z = \mathbb{E}_{z_0} \int_0^\infty \mathbb{1}_{\mathrm{d}z}(Z_u)\mathrm{d}L_u^i$$

- **Quantity of time** spend in a set

Goal ♯3

Study g and h_i

Operators associated to Green's functions

$Gf(z_0) = \mathbb{E}_{z_0} \int_0^\infty f(Z_t)\mathrm{d}t = \int_{\mathbb{R}_+^2} f(z)g^{z_0}(z)\mathrm{d}z$

$H_i\phi_i(z_0) = \mathbb{E}_{z_0} \int_0^\infty \phi_i(Z_t)\mathrm{d}L_t^i = \int_{\mathbb{R}_+^2} \phi_i(z)h_i^{z_0}(z)\mathrm{d}z$

PDE with oblique Neumann conditions

PDE : Green's functions

Green's functions are fundamental solutions of PDE with **oblique Neumann conditions** in the quadrant :

$$\begin{cases} \mathcal{L}u = -f & \text{in the quadrant où } \mathcal{L}f = \frac{1}{2}\nabla \cdot \Sigma\nabla + \mu \cdot \nabla \\ \partial_{R_i} u = \phi_i & \text{on the boundary where } \partial_{R_i} = R^i \cdot \nabla \end{cases}$$

Solution : $$u = Gf + H_1\phi_1 + H_2\phi_2$$

Absorption probability p_A

$$\begin{cases} \mathcal{L}p_A = 0 \\ \partial_{R_i} p_A = 0 \end{cases} \text{ et } \begin{cases} p_A(0) = 1 \\ \lim_{\infty} p_A = 0 \end{cases}$$

Escape probability p_1 (abscissa)

$$\begin{cases} \mathcal{L}p_1 = 0 \\ \partial_{R_i} p_1 = 0 \end{cases} \text{ et } \begin{cases} \lim_{u\infty} p_1(u,0) = 1 \\ \lim_{v\infty} p_1(0,v) = 0 \end{cases}$$

Plan

Key steps

Analytic approach

- Find a **functional equation**
- Study the **kernel** (Riemann surface, group)
- **Continue** analytically generating functions
- Establish a **boundary value problem**
- Solve it to *determine an explicit formula* (Sokhotski–Plemelj, invariants)
- *Find the asymptotics* (singularities, transfer lemmas, saddle point method)

Laplace transforms

- **Discret case :** generating functions of Green's functions $g_{i,j}^{i_0,j_0}$ on $\mathbb{Z}_+^2$ is the **generating series** $\sum_{\mathbb{Z}_+^2} g_{i,j}^{i_0,j_0} x^i y^j$.
- **Continuous case :**
  - **Laplace transform** of the **Green's function**

$$L(x,y) = \iint_{\mathbb{R}_+^2} e^{xu+yv} g^{z_0}(u,v)\mathrm{d}u\mathrm{d}v$$

  - On the *boundaries* we define

$$L_2(x) = \int_{\mathbb{R}_+} e^{xu} h_2^{z_0}(u)\mathrm{d}u, \quad L_1(y) = \int_{\mathbb{R}_+} e^{yv} h_1^{z_0}(v)\mathrm{d}v$$

- Similarly, we define the Laplace transform :
  - $\widetilde{L}$, $\widetilde{L}_1$, $\widetilde{L}_2$ of the **absorption probability** p_A,
  - $\widehat{L}$, $\widehat{L}_1$, $\widehat{L}_2$ of the **escape probability** along the abscissa p_1.

Functional equation

This equation binds together the Laplace transform of Green's functions.

Functional equation (2020, F.)

$$-K(x,y)L(x,y) = K_1(x,y)L_1(y) + K_2(x,y)L_2(x) + e^{(x,y)\cdot z_0}$$

where

$$\begin{cases} K(x,y) = \frac{1}{2}(\sigma_{11}x^2 + \sigma_{22}y^2 + 2\sigma_{12}xy) + \mu_1 x + \mu_2 y, \\ K_1(x,y) = x + r_1 y, \\ K_2(x,y) = r_2 x + y. \end{cases}$$

- Connect what happens in the quadrant and on the boundaries.
- The function K is called the kernel.
- The term $e^{(x,y)\cdot z_0}$ contains the dependence to the starting z_0.

Proof : Itô formula, PDE.

Functional equations

Green's functions (2020, F.)

$$-K(x,y)L(x,y) = K_1(x,y)L_1(y) + K_2(x,y)L_2(x) + e^{(x,y)\cdot z_0}$$

Absorption probability (2021, Ernst, F.)

$$-K(x,y)\widetilde{L}(x,y) = \widetilde{K}_1(x,y)\widetilde{L}_1(y) + \widetilde{K}_2(x,y)\widetilde{L}_2(x)$$

Escape probability (2020+, Fomichov, F., Ivanovs)

$$-K(x,y)\widehat{L}(x,y) = \widetilde{K}_1(x,y)\widehat{L}_1(y) + \widetilde{K}_2(x,y)\widehat{L}_2(x) + cp_1(0)$$

$$K(x,y) = \frac{1}{2}(\sigma_{11}x^2 + \sigma_{22}y^2 + 2\sigma_{12}xy) + \mu_1 x + \mu_2 y,$$

$$K_1(x,y) = x + r_1 y, \quad K_2(x,y) = r_2 x + y,$$

$$\widetilde{K}_1(x,y) = \frac{r_2 x + y}{2} + \sigma_{12}x + \mu_2, \; \widetilde{K}_2(x,y) = \frac{r_1 y + x}{2} + \sigma_{12}y + \mu_1.$$

Proof : Itô/Dynkin formula, IPP and PDE.

What is a boundary value problem ?

A boundary value problem is made of two conditions :

- a **regularity condition** on a set
- a **boundary condition**

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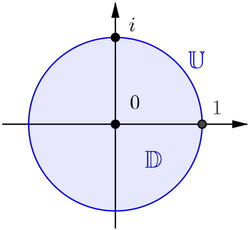
Example :

1. I is meromorphic on the unit disc $\mathbb{D}$ has one single poleor order one in 0
2. $\boxed{I(\bar{x}) = I(x)}$ for $x \in \mathbb{U}$
   unit circle



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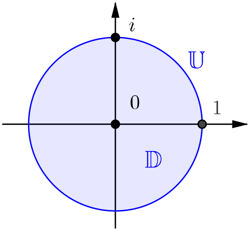


The solution (up to constants) is $\boxed{I(x) = x + \frac{1}{x}}$.

I is a **conformal gluing function** which unite the upper and the lower part of $\mathbb{U}$.

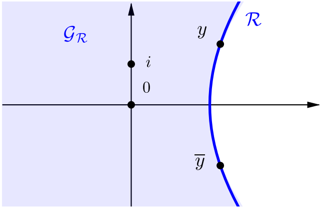
I is an **invariant for the conjugation** on the boundary $\mathbb{U}$.

Boundary value problem (2020, F.)

1. L_1 is analytic on $\mathcal{G}_{\mathcal{R}}$ and tends to 0 at infinity ;
2. L_1 satisfy the boundary condition

$$L_1(\overline{y}) = G(y)L_1(y) + g(y), \qquad \forall y \in \mathcal{R}.$$

- G and g depend of parameters
- $\mathcal{R}$ is an **hyperbola** defined by the kernel



Boundary value problems

Green's functions (2020, F.)

$$\boxed{L_1(\overline{y}) = G(y)L_1(y) + g(y), \qquad \forall y \in \mathcal{R}.}$$

Absorption probability (2021, Ernst, F.)

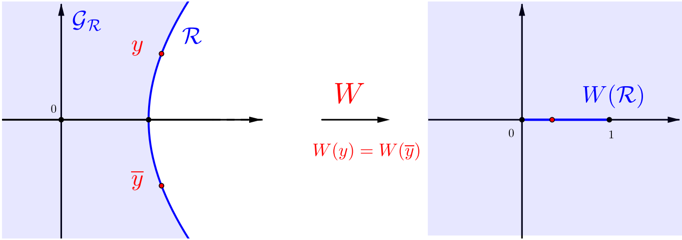
$$\boxed{\widetilde{L}_1(\overline{y}) = \widetilde{G}(y)\widetilde{L}_1(y), \qquad \forall y \in \mathcal{R}.}$$

Escape probability (2020, Fomichov, F., Ivanovs)

$$\boxed{\widehat{L}_1(\overline{y}) = \widehat{G}(y)\widehat{L}_1(y) + \widehat{g}(y), \qquad \forall y \in \mathcal{R}.}$$

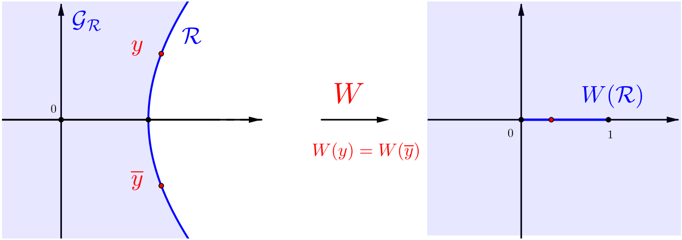
Preuve : Cancel the kernel in the functional equation.

- Conformal gluing function W unite the upper and the lower part of the hyperbola $\mathcal{R}$



$$W(y) = T_{\frac{\pi}{\beta}}\left(- \frac{2y - (y^+ + y^-)}{y^+ - y^-} \right)$$

- $\beta = \arccos\left(\frac{-\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}\right)$ in the angle of the cone
- $T_{\frac{\pi}{\beta}}$ is a **generalized Tchebychev polynomial**
  $T_{\frac{\pi}{\beta}}(x) = \cos(\frac{\pi}{\beta}\arccos(x)) = \frac{1}{2}\left\{\left(x+\sqrt{x^2-1}\right)^{\frac{\pi}{\beta}} + \left(x-\sqrt{x^2-1}\right)^{\frac{\pi}{\beta}}\right\}$



Carleman boundary value problem $\longrightarrow$ **Riemann** boundary value problem

- $M := \widetilde{L}_1 \circ W^{-1}$ and
- M^+ and M^- the upper and lower limit M on $[0, 1]$
- $H := \widetilde{G} \circ W^{-1}$

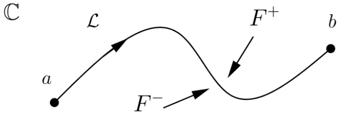
▶ $\widetilde{L}_1(\overline{y}) = \widetilde{G}(y)\widetilde{L}_1(y), \forall y \in \mathcal{R}$

▶ $M^+(t) = H^-(t)M^-(t), \forall t \in [0, 1]$

Sokhotski–Plemelj formulae

We define for $z \in \mathbb{C} \setminus \mathcal{L}$

$$F(z) := \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{f(t)}{t-z} \, \mathrm{d}t$$



- F is sectionaly analytic on $\mathbb{C} \setminus \mathcal{L}$
- F^+ and F^- are limits of F on both parts of $\mathcal{L}$

Sokhotski–Plemelj formula

$$F^+(t) - F^-(t) = f(t)$$

$M = e^F$ and $f = \ln H \Rightarrow M^+ = H^- M^-$

Solution of the Riemann boundary value problem

- If $M^+ = H^- M^-$ then

$$M = \exp \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{\ln H(t)}{t-z} \, \mathrm{d}t$$

Explicit expression

Absorption probability (2021, Ernst, F.)

$$\widetilde{L}_1(y) = \frac{W'(0)}{W(y) - W(0)} \left(\frac{W(0) - W(p)}{W(y) - W(p)} \right)^{-\chi} \exp\left\{ \frac{1}{2i\pi} \int_{\mathcal{R}^-} \log G(t) \frac{W'(t)}{W(t) - W(y)} \mathrm{d}t \right\}$$

- Inversion of the Laplace transform $\widetilde{L}_1$

$\hookrightarrow$ absorption probability $p_A(z_0)$

Escape probability and Green's functions

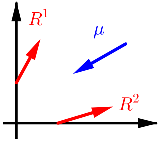
- Boundary value problems satisfied by L_1 and $\widehat{L}_1$ are **non homogeneous**
  $\hookrightarrow$ More complicated formulas

Escape probability along abscissa

(2020+, Fomichov, F., Ivanovs)

Starting from the origin and with identity covariance :

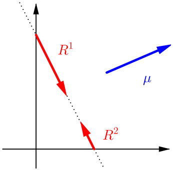
$$p_1(0) = \frac{r_1(r_2\mu_2 - \mu_1)(\mu_2 + r_2\mu_1)}{\mu_1\mu_2(r_1 r_2 - 1)(r_1 + r_2)}.$$



Dual skew symmetry (2021, Ernst, F.)

Following statements are equivalent :

1. Exponential absorption probability, i.e. $\exists v \in \mathbb{R}^2$ such that $\boxed{p_A(z_0) = e^{z_0 \cdot v}}$
2. Reflection vectors are opposite, i.e. $r_{11}r_{22} - r_{21}r_{12} = 0$

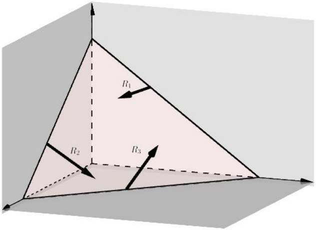


Remarkable cases

Dual skew symmetry in an orthant (2022, F., Raschel)

Following statements are equivalent :

1. Absorption probability of product form
   i.e. there exist $f_1, \cdots, f_n$ such that $f(z) = f_1(z_1) \cdots f_n(z_n)$
2. Exponential absorption probability
   i.e. $\exists v \in \mathbb{R}^n$ tel que $\boxed{p_A(z) = e^{z \cdot v}}$
3. Reflection vectors are coplanar
   i.e. $\det R = 0$

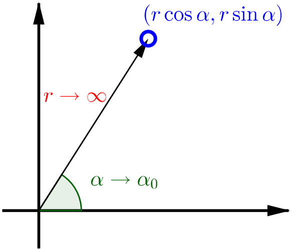


Plan

Asymptotics

▶ Bivariate asymptotics of Green's functions along each directions

$$g^{z_0}(r\cos\alpha, r\sin\alpha) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} \ ?$$



Method :

- **Transfer lemma** (study of **singularities**)
- **Saddle point method** (inversion of Laplace transform)

Transfer lemmas and sadlle point method

Transfer lemma

- L_1 Laplace transform of h_1
- a the smallest singularity and has order k

$$L_1(z) \sim \frac{c}{(a-z)^k}$$

▶ Then,

$$h_1(x) \underset{x\to\infty}{\sim} \frac{c}{\Gamma(k)} x^{k-1} e^{-ax}$$

Saddle point method

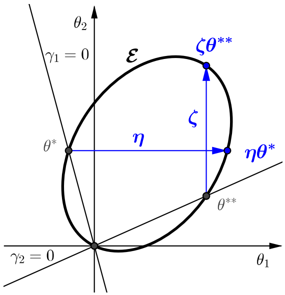
- f and g holomorphic
- a critic point
  $f'(a) = 0$ et $f''(a) \neq 0$

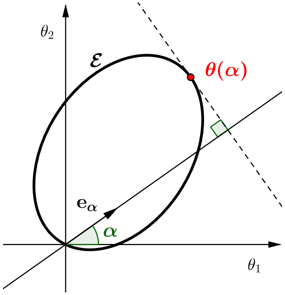
▶ Then, for a contour $\mathcal{C}$

$$\int_{\mathcal{C}} g(z) e^{xf(z)} \mathrm{d}z \underset{x\to\infty}{\sim} g(a)\sqrt{\frac{2\pi}{-f''(a)}} x^{-\frac{1}{2}} e^{-ax}$$

Poles and saddle point

$\mathcal{E} := \{(x,y) \in \mathbb{R}^2 : K(x,y) = 0\}$ **ellipse**





Poles $\eta\theta^*$ et $\zeta\theta^{**}$

Saddle point $\theta(\alpha)$

Analytic combinatorics to several variables

Asymptotis of Green's functions (2021, Ernst, F.)

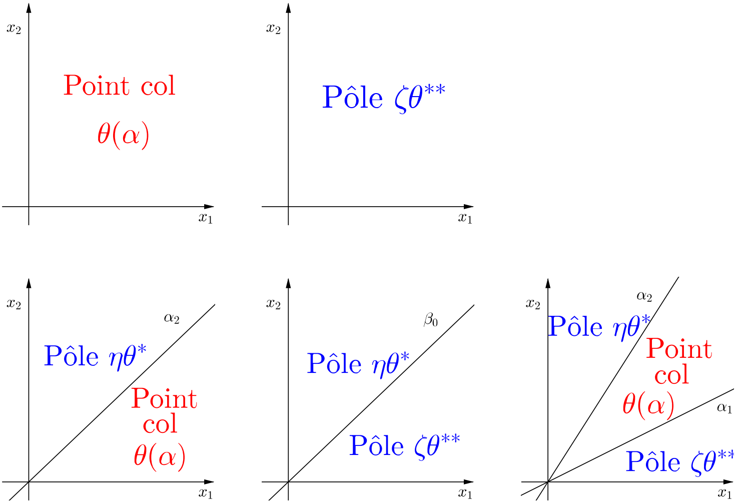
Let $\alpha_0 \in (0, \pi/2)$. We have

$$g^{z_0}(r\cos\alpha, r\sin\alpha) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} C r^{\kappa} e^{-r(\cos\alpha,\sin\alpha)\cdot\tau(\alpha)}$$

where :

- Critical exponent κ equal to $-3/2$, $-1/2$, 0 or 1.
- The decay rate $\tau(\alpha) \in \mathbb{R}^2$ comes from poles or from the saddle point. That is : $\eta\theta^*$, $\zeta\theta^{**}$ or $\theta(\alpha)$.

Asymptotics according to the direction : several cases



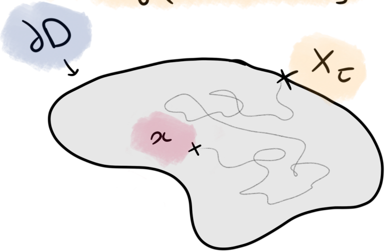
Heuristic of Martin and Poisson boundary

X_t = Brownien absorbé

D = Domaine borné

x = point de départ

$\tau = \inf\{t : X_t \in \partial D\}$



$$\begin{cases} \Delta u = 0 & \text{Harmonique} \\ u\big|_{\partial D} = f & \text{Dirichlet} \end{cases}$$

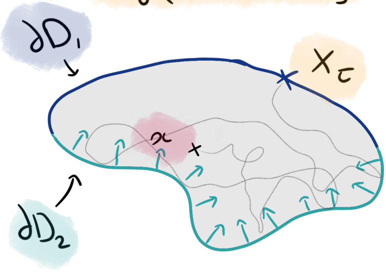
$$u(x) = \mathbb{E}_x\left[f(X_\tau)\right] = \int_{\partial D} f(y)\, \underbrace{\partial g^x(y)\, dy}_{\mathbb{P}(X_\tau \in dy)}$$

X_t = Brownien absorbé en ∂D_1, réfléchi en ∂D_2

D = Domaine borné

x = point de départ

$\tau = \inf\{t : X_t \in \partial D_1\}$



$$\begin{cases} \Delta u = 0 & \text{Harmonique} \\ u\big|_{\partial D_1} = f & \text{Dirichlet} \\ \partial u\big|_{\partial D_2} = 0 & \text{Neumann} \end{cases}$$

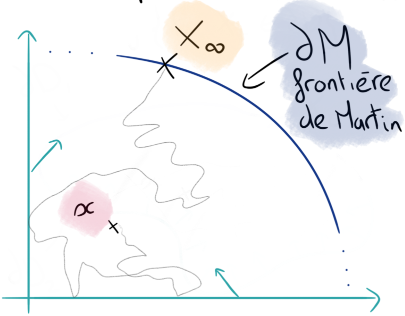
$$u(x) = \mathbb{E}_x\left[f(X_\tau)\right] = \int_{\partial D_1} f(y) \underbrace{\partial g^x(y) dy}_{\mathbb{P}(X_\tau \in dy)}$$

X_t = transient

Quadrant <u>non borné</u>

M = compactification de Martin

$$\begin{cases} \Delta u = 0 & \text{Harmonique} \\ u\big|_{\partial M} = f & \text{Dirichlet} \\ \partial u\big|_{\text{axes}} = 0 & \text{Neumann} \end{cases}$$



$$u(x) = \mathbb{E}_x\left[f(X_\infty)\right]$$

$$= \int_{\partial M} f(y) \underbrace{k^x(y)\,dy}_{\mathbb{P}(X_\infty \in dy)}$$

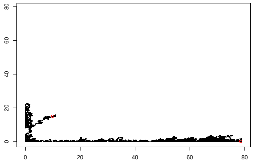
Noyau de Martin

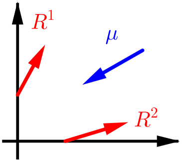
Harmonic functions : probabilistic interpretations

▶ Trivial Martin boundary (**one point**) ⇔ constants are the only harmonic functions

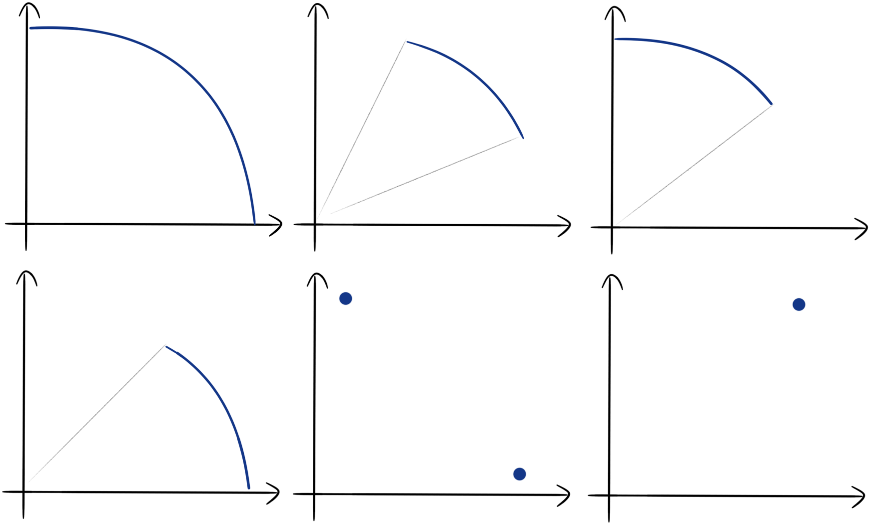
Transient reflected Brownian motion in the quadrant

▶ **Two points** Martin boundary ⇔ Escape probability p_1 and p_2 are the **only** harmonic functions ($p_1 + p_2 = 1$)





Martin boundary in the quadrant



Thank you for your attention !

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